Final Exam

7:00-10:00pm, 10 May 2024

Your First Name: **Your Last Name:**

SIGN Your Name: **Your SID Number:**

Your Exam Room:

Name of Person Sitting on Your Left:

Name of Person Sitting on Your Right:

Name of Person Sitting in Front of You:

Name of Person Sitting Behind You:

Instructions:

(a) *As soon as the exam starts,* **please write your student ID in the space provided at the top of every page!** *(We will remove the staple when scanning your exam.)*

(b) *There are 17 pages (2-sided) on the exam. Notify a proctor immediately if a page is missing.*

(c) *We will* **not** *grade anything outside of the space provided for a question (i.e., either a designated box if it is provided, or otherwise the white space immediately below the question).* **Be sure to write your full answer in the box or space provided!** *Scratch paper is provided on request; however, please bear in mind that nothing you write on scratch paper will be graded!*

(d) **The questions vary in difficulty, so if you get stuck on any question you are strongly advised to leave it and return to it later. In particular, you may find some of the later questions easier than some of the earlier ones.** *There is no penalty for incorrect answers.*

(e) *In the interests of fairness, proctors will* **not** *answer clarifying questions during the exam. If you believe there is an ambiguity or error in a problem, write down what you think is the most appropriate response and move on. Any legitimate such issues will be handled during grading.*

(f) *You may consult* **2 two-sided "cheat sheets"** *of notes. Apart from that, you may not look at any other materials. Calculators, phones, computers, and other electronic devices are* **not** *permitted.*

(g) *You may use, without proof, theorems and lemmas that were proved in the notes and/or in lecture.*

(h) *You have 180 minutes: there are 14 questions on this exam worth a total of 140 points.*

[exam starts on next page]

0. Academic Honesty Pledge:

Berkeley Honor Code: As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others. In particular, I acknowledge that:

- I alone am taking this exam. Other than with the instructor and course staff, I will not have any verbal, written, or electronic communication about the exam with anyone while I am taking the exam or while others are taking the exam.
- I will not refer to any books, notes, or online sources of information while taking the exam, except as explicitly allowed by the instructor.
- I will not take screenshots, photos or otherwise make copies of exam questions to share with others.

Signed:

[exam continues on next page]

1. Multiple Choice [21 points].

Answer by shading the **single** *correct bubble in each case; no justifications; no penalty for incorrect answers.*

(a) TRUE or FALSE: Suppose $\forall x \geq 0\,(\neg P(x))$. Then, $\forall x \geq 0\,\big(P(x) \implies (\exists y\; y \geq x \wedge Q(y))\big)$. *2pts*

◯ TRUE ◯ FALSE

(b) Recall that a propositional form is a statement that combines propositional variables with standard connectives $\wedge$, $\vee$, and $\neg$. For this problem, assume there are no quantifiers. *2pts*
The number of unique (up to logical equivalence) propositional forms on n propositional variables $P_1, P_2, \ldots, P_n$ is

◯ n ◯ $2n$ ◯ n^2 ◯ 2^n ◯ 2^{2^n}

(c) Let n and p be integers where p is an odd prime. If $n \equiv -n \pmod{p}$ then $n \pmod{p}$ must be *2pts*

◯ 0 ◯ 1 ◯ 2 ◯ $p/2$ ◯ $p-1$

(d) TRUE or FALSE: There exists a computer program which takes as inputs a program P, an input x for P, and a line number n, and determines whether the nth line of code in P is executed when P is run on input x. *2pts*

◯ TRUE ◯ FALSE

(e) The number of polynomials of degree at most k with integer coefficients (in $\mathbb{Z}$) is *2pts*

◯ Finite ◯ Countably Infinite ◯ Uncountably Infinite

(f) Suppose we wish to send a message comprising 10 packets, each of which is an integer between 0 and 15, over a channel which corrupts up to 6 packets. Using the Reed-Solomon Code, the polynomial we construct must be over $GF(p)$ where the minimum value of prime p is *2pts*

◯ 11 ◯ 17 ◯ 23 ◯ 29

(g) Let G be an undirected graph on n vertices that consists of k trees on disjoint sets of vertices. The number of edges in G is *2pts*

◯ $n-1$ ◯ $k(n-1)$ ◯ $n-k$ ◯ $n-k+1$ ◯ not determined

(h) If n, m, k are positive integers where $k \leq n, k \leq m$, then $\sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i}$ is equal to *2pts*

○ $\binom{n+m}{k}$ ○ $\binom{nm}{k}$ ○ 2^{n+m} ○ $\frac{n!m!}{(k!)^2}$ ○ none of these

(i) Let $u = 0000000$ and $v = 1011001$ be two vertices in the 7-dimensional hypercube. The number of *shortest* (in the sense of having fewest edges) paths between u and v is *2pts*

○ 1 ○ 4 ○ 16 ○ 24 ○ 128

(j) Suppose an undirected simple graph (i.e., no self-loops or multiple edges) has n vertices and m edges. For which of the following combinations of n and m is it **possible** that the graph contains an Eulerian tour? For each combination choose TRUE if such a tour **may** exist, and FALSE if such a tour **cannot** exist.

TRUE FALSE

○ ○ $n = 7,\ m = 7$ *1pt*

○ ○ $n = 4,\ m = 5$ *1pt*

○ ○ $n = 5,\ m = 7$ *1pt*

[exam continues on next page]

2. Short Answers [9 points].

(a) The value of $3^{42} \pmod{55}$ is *2pts*

Write your answer in the box; no explanation needed.

(b) If $17x \equiv 7 \pmod{42}$, then x is *2pts*

Write your answer in the box; no explanation needed.

(c) Suppose we are using the standard polynomial secret sharing scheme over $GF(5)$ and $k = 3$ shares suffice to reconstruct the secret $P(0)$. Let the shares be $P(1) = 1$, $P(2) = 1$, and $P(4) = 2$. Then the value of the secret is *3pts*

Write your answer in the box; no explanation needed.

(d) Consider the function $f(x) = x^2 \pmod{7}$ mapping $\{0, 1, 2, 3, 4, 5, 6\}$ to $\{0, 1, 2, 3, 4, 5, 6\}$. *2pts*
Is f a bijection? If yes, give its inverse. If not, give values of $x, f(x)$ justifying why.

Write your answer by shading the appropriate circle and writing in the box only; no further explanation needed.

◯ Bijection Inverse:

OR

◯ Not a bijection Values of $x, f(x)$:

3. Proofs [10 points].

(a) Prove by induction that, for any $x \neq 1$, the following identity holds for all $n \geq 1$. *5pts*

$$\frac{x^n - 1}{x - 1} = \sum_{i=0}^{n-1} x^i$$

Clearly label your base case and induction step.
(**Note:** No points will be awarded for proofs that do not use induction!)

(b) Prove that if $2^m - 1$ is prime for some natural number m, then m is also prime. [HINT: Try a proof by contraposition, and use the previous part.] *5pts*

[exam continues on next page]

4. Stable Matching [6 points]

Consider the following sets of matching preferences for four jobs 1, 2, 3, 4 and four candidates A, B, C, D.

Job	Preferences
1	$A > B > C > D$
2	$B > C > D > A$
3	$C > D > A > B$
4	$B > A > C > D$

Candidate	Preferences
A	$2 > 3 > 1 > 4$
B	$3 > 1 > 2 > 4$
C	$1 > 2 > 4 > 3$
D	$2 > 4 > 3 > 1$

For each of the matchings in (a) and (b) below, indicate whether the matching is stable or not stable. If it is stable, indicate whether it is job-optimal or candidate-optimal. If it is not stable, identify a rogue couple.

Shade the appropriate circle to indicate your answer; if applicable, write the rogue couple in the box given. No explanation needed.

(a) $\{(1, B), (2, C), (3, A), (4, D)\}$. *3pts*

◯ Stable

◯ Job-Optimal

◯ Candidate-Optimal

OR

◯ Not Stable

Rogue Couple: ☐

(b) $\{(1, A), (2, B), (3, C), (4, D)\}$. *3pts*

◯ Stable

◯ Job-Optimal

◯ Candidate-Optimal

OR

◯ Not Stable

Rogue Couple: ☐

5. [4 points] Suppose there is a bag containing two fair four-sided dice, one with the numbers $1, 2, 3, 4$ on its faces and the other with the numbers $1, 2, 2, 4$ on its faces. A die is randomly drawn from the bag and rolled twice.

(a) Given that the first roll is a 3, what is the probability that the second roll is a 2? *2pts*

Write your answer in the box; no explanation needed.

(b) Given that the first roll is a 2, what is the probability that the second roll is a 2? *2pts*

Write your answer in the box; no explanation needed.

6. [6 points] Let P and Q be polynomials of degree at most d, whose coefficients are chosen uniformly and independently at random from $GF(p)$ for a prime $p > d$.

(a) What is $\mathbb{P}[P = Q]$, where $P = Q$ denotes the fact that $P(x) \equiv Q(x) \pmod{p}$ for all $x \in GF(p)$? *2pts*

Write your answer in the box; no explanation needed.

(b) What is $\mathbb{P}[P(0) \equiv Q(0) \pmod{p}]$? *2pts*

Write your answer in the box; no explanation needed.

(c) Given an integer $k \in \{0, \ldots, d-1\}$, what is $\mathbb{P}[P(i) \equiv Q(i) \pmod{p}$ for all $i \in \{0, \ldots, k\}]$? *2pts*

Write your answer in the box; no explanation needed.

7. [4 points] You are building a cloud computing startup and you have N compute nodes. The failure time for a single node is given by an exponential random variable with parameter λ and is independent of other nodes. You choose to distribute these nodes across ℓ locations, so that each location gets the same number of nodes $k = N/\ell$ (assume that ℓ divides N). A location fails if *some node* at that location fails. The whole system fails if *all locations* fail. *4pts*

What is the probability that the system fails before time t?

Write your answer in the box; no explanation needed.

8. [8 points] The continuous random variable X has pdf

$$f(x) = \begin{cases} \alpha x & \text{for } 0 \le x \le 1; \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the value of α. *2pts*

Write your answer in the box; no explanation needed.

(b) Compute $\mathbb{P}\left[X \le \frac{1}{2}\right]$. **[Note: Write your answer in terms of α.]** *2pts*

Write your answer in the box; no explanation needed.

(c) Compute $\mathbb{E}[X]$. **[Note: Write your answer in terms of α.]** *2pts*

Write your answer in the box; no explanation needed.

(d) Compute Var (X). **[Note: Write your answer in terms of α and μ, where $\mu = \mathbb{E}[X]$.]** *2pts*

Write your answer in the box; no explanation needed.

9. [8 points] Let X, Y, Z be random variables with the following properties:

- X, Y are independent and Y, Z are independent
- $\mathbb{E}[X] = 0$; $\mathbb{E}[Y] = 1$; $\mathbb{E}[Z] = 2$.
- $\text{Var}(X) = \text{Var}(Y) = \text{Var}(Z) = 10$.

For each of the following statements, determine if it **must** be true, **must** be false, or is undetermined (i.e., may be either true or false). Shade one bubble for each statement.

TRUE	FALSE	UNDET.		
○	○	○	X, Z are independent.	*1pt*
○	○	○	$\mathbb{E}[X + 2Y - Z] = 0$.	*1pt*
○	○	○	$\mathbb{P}[Y \geq 10] \leq \frac{1}{10}$.	*1pt*
○	○	○	$\mathbb{E}[XY] = 2$.	*1pt*
○	○	○	$\text{Var}(X - Y) = 20$.	*1pt*
○	○	○	$\text{Var}(2X) = 20$.	*1pt*
○	○	○	$\mathbb{P}[\lvert X\rvert \geq 5] \leq \frac{2}{5}$.	*1pt*
○	○	○	$\mathbb{P}\left[\lvert X\rvert \geq \sqrt{10}\right] > 0$.	*1pt*

10. [2 points] Let $X_1, \ldots, X_n$ be i.i.d. random variables that are uniform over the set of values $\{1, 2, 3\}$, and let $S_n = X_1 + \cdots X_n$. Determine parameters a and b such that the random variable $\frac{S_n - a}{b}$ converges to the standard normal distribution $\mathcal{N}(0, 1)$ as $n \to \infty$. *2pts*

Write your answer in the boxes; no explanation needed.

$a =$ ☐ $b =$ ☐

11. Musical Counting [18 points].

In preparation for the CS70 musical, we want to understand the various ways we can arrange and combine musical notes. Throughout this problem, we will use the following terminology and assumptions. (Note that this may deviate from standard musical practice: you should disregard any prior musical knowledge you have for this problem!)

- A *piano* has 88 keys in total, 52 of which are white and the other 36 black.
- A *chord* is any set of k distinct keys played simultaneously (i.e., order does not matter), where $2 \leq k \leq 10$.
- A *melody* is a sequence of any number of individual keys played one after the other (here order matters, and keys can repeat unless stated otherwise).

Write your answers in the boxes provided. Your answers should not include any summations. Leave your answers in terms of binomial coefficients (if there are any) without simplifying them. No justifications required.

(a) How many chords of $k = 5$ keys containing exactly 2 white keys and 3 black keys are there? *3pts*

(b) Given a fixed set of 10 distinct keys, how many chords can be played from these keys? *3pts*

(c) How many melodies consisting of exactly 12 keys (repetitions allowed) are there? *3pts*

(d) How many melodies are there that consist of 40 white keys and 10 black keys (with repetitions allowed), such that no two black keys are played one after the other (i.e., there has to be at least one white key between any two black keys)? *3pts*

(e) How many ways are there to choose 200 keys? Each of the 88 piano keys can be used any number of times (or not at all), and the order does not matter. *3pts*

(f) Suppose now that you are given a melody of 200 *ordered* keys. In how many ways can this melody be split into exactly 4 consecutive parts, such that each part contains at least 20 keys? *3pts*

12. Isolated Vertices on a Cycle Network [16 points]

Suppose we have a network of $n \geq 3$ processors connected in the form of a single cycle (see the left-hand figure below for the example $n = 5$). Each connection (edge) in the network fails and is removed independently with probability $\frac{1}{2}$. We call a processor (vertex) *isolated* if it has no remaining edges incident to it. (In the $n = 5$ example in the right-hand figure below, three edges have failed and only vertex 2 is isolated.)

(a) For any vertex i, what is $\mathbb{P}[i \text{ is isolated}]$? *2pts*

Write your answer in the box; no explanation needed.

(b) Let X denote the number of isolated vertices. Compute the expectation $\mathbb{E}[X]$. *2pts*

Write your answer in the box; no explanation needed.

(c) For two *adjacent* vertices i, j in the cycle, what is $\mathbb{P}[i \text{ and } j \text{ are both isolated}]$? *2pts*

Write your answer in the box and give a very brief explanation in this space.

(d) For two *non-adjacent* vertices i, j, what is $\mathbb{P}[i \text{ and } j \text{ are both isolated}]$? *2pts*

Write your answer in the box and give a very brief explanation in this space.

(e) Compute Var (X). [**Note**: You will need to use your answers to parts (b), (c) and (d). Indicate clearly where you use these answers!] *4pts*

Write your answer in the box and show your working in the space provided.

(f) Use Chebyshev's inequality to get an upper bound on the probability $\mathbb{P}\left[X \geq \frac{n}{2}\right]$ that there are at least $\frac{n}{2}$ isolated vertices. [**Note:** You will need to use your answers to parts (b) and (e). Indicate clearly where you use these answers!] *4pts*

Write your answer in the box and show your working in the space provided.

13. Dolphin Watching [16 points]

You are going on a boat tour to observe dolphins off the Northern California coast. You are given the following information about the dolphins' behavior:

- On average, a dolphin will approach your boat every 20 minutes.
- When a dolphin approaches your boat, it will swim along with you. For each minute until it leaves, it will perform a trick with probability p, otherwise do nothing.
- At the end of each minute, a dolphin at the boat will leave with probability q, otherwise it will stay for at least another minute.
- All dolphin arrivals and all dolphin behaviors are mutually independent.

Let's do some dolphin stats. [**Note: Your answers may include binomial coefficients and fractions, but should not include any summations.**]

(a) Let the r.v. N denote the number of dolphins that approach your boat **per hour**. What is the distribution of N? (Give its name and parameter.) *2pts*

Write your answer in the box; no explanation needed.

(b) As the i-th dolphin approaches your boat, let X_i denote the number of minutes the dolphin will swim alongside your boat. What is the distribution of X_i? (Give its name and parameter.) *2pts*

Write your answer in the box; no explanation needed.

(c) What is the probability that dolphin i will swim alongside your boat for exactly 5 minutes? *2pts*

Write your answer in the box; no explanation needed.

(d) How long will a dolphin swim alongside your boat on average? *2pts*

Write your answer in the box; no explanation needed.

(e) Let Z_i denote the number of tricks that dolphin i will perform while swimming alongside your boat. Compute the probability that a dolphin will perform exactly k tricks, given that it swims alongside for exactly 20 minutes. *2pts*

Write your answer in the box; no explanation needed.

(f) Given that a dolphin swims alongside your boat for 20 minutes, what is the expected number of tricks it performs? *2pts*

Write your answer in the box; no explanation needed.

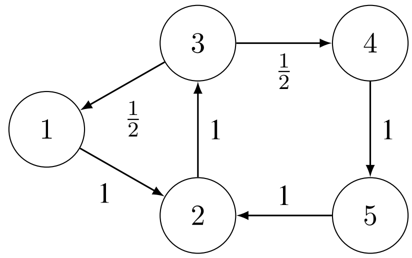
(g) Compute $\mathbb{E}[Z_i]$. *4pts*

Write your answer in the box and show your working in the space provided.

[exam continues on next page]

14. Five-State Markov Chain [12 points].

Consider the following Markov chain with states $1, \ldots, 5$.



(a) Is this chain irreducible? Briefly explain your answer. *2pts*

Write your answer in the box.

(b) Is this chain aperiodic? Briefly explain your answer. *2pts*

Write your answer in the box.

(c) Let π denote the invariant distribution of the chain. Given that $\pi(1) = \pi(5) = \frac{1}{7}$, compute $\pi(2)$. *3pts*

Write your answer in the box; no explanation needed.

(d) For $i = 1, \ldots, 5$, let $\beta(i)$ denote the expected number of steps to reach state 1 starting from state i. Given that $\beta(3) = 5$, compute $\beta(5)$. *3pts*

Write your answer in the box; no explanation needed.

(e) Suppose the transition from state 4 to state 5 is removed and replaced by a self-loop with probability 1 on state 4. What is now the expected number of steps to reach state 1 starting from state 5? *2pts*

Write your answer in the box; no explanation needed.