

## 1 Contraposition

Prove the statement "if  $a + b < c + d$ , then  $a < c$  or  $b < d$ ".

### Solution:

The implication we're trying to prove is  $(a + b < c + d) \implies ((a < c) \vee (b < d))$ , so the contrapositive is  $((a \geq c) \wedge (b \geq d)) \implies (a + b \geq c + d)$ . The proof of this is quite straightforward: since we have both that  $a \geq c$  and that  $b \geq d$ , we can just add these two inequalities together, giving us  $a + b \geq c + d$ , which is exactly what we wanted.

## 2 Perfect Square

A *perfect square* is an integer  $n$  of the form  $n = m^2$  for some integer  $m$ . Prove that every odd perfect square is of the form  $8k + 1$  for some integer  $k$ .

### Solution:

We will proceed with a direct proof. Let  $n = m^2$  for some integer  $m$ . Since  $n$  is odd,  $m$  is also odd, i.e., of the form  $m = 2l + 1$  for some integer  $l$ . Then,  $m^2 = 4l^2 + 4l + 1 = 4l(l + 1) + 1$ . Since one of  $l$  and  $l + 1$  must be even,  $l(l + 1)$  is of the form  $2k$  for some integer  $k$  and  $n = m^2 = 8k + 1$ .

## 3 Numbers of Friends

Prove that if there are  $n \geq 2$  people at a party, then at least 2 of them have the same number of friends at the party.

(Hint: The Pigeonhole Principle states that if  $n$  items are placed in  $m$  containers, where  $n > m$ , at least one container must contain more than one item. You may use this without proof.)

### Solution:

We will prove this by contradiction. Suppose the contrary that everyone has a different number of friends at the party. Since the number of friends that each person can have ranges from 0 to  $n - 1$ , we conclude that for every  $i \in \{0, 1, \dots, n - 1\}$ , there is exactly one person who has exactly  $i$  friends at the party. In particular, there is one person who has  $n - 1$  friends (i.e., friends with everyone), and there is one person who has 0 friends (i.e., friends with no one), which is a contradiction.

Here, we used the pigeonhole principle because assuming for contradiction that everyone has a different number of friends gives rise to  $n$  possible containers. Each container denotes the number of friends that a person has, so the containers can be labelled  $0, 1, \dots, n - 1$ . The objects assigned to these containers are the people at the party. However, containers 0,  $n - 1$  or both must be empty

since these two containers cannot be occupied at the same time. This means that we are assigning  $n$  people to at most  $n - 1$  containers, and by the pigeonhole principle, at least one of the  $n - 1$  containers has to have two or more objects i.e. at least two people have to have the same number of friends.

## 4 Fermat's Contradiction

Prove that  $2^{1/n}$  is not rational for any integer  $n \geq 3$ . (*Hint*: Use Fermat's Last Theorem. It states that there exists no positive integers  $a, b, c$  s.t.  $a^n + b^n = c^n$  for  $n \geq 3$ .)

### **Solution:**

If not, then there exists an integer  $n \geq 3$  such that  $2^{1/n} = \frac{p}{q}$  where  $p, q$  are positive integers. Thus,  $2q^n = p^n$ , and this implies

$$q^n + q^n = p^n,$$

which is a contradiction to the Fermat's Last Theorem.

## 5 Prime Form

Prove that every prime number  $m > 3$  is either of the form  $6k + 1$  or  $6k - 1$  for some integer  $k$ .

### **Solution:**

First we note that any integer can be written in one of the forms  $6k, 6k + 1, 6k + 2, 6k + 3, 6k + 4$  and  $6k + 5$ . (Note that  $6k + 5$  is equal to  $6(k + 1) - 1$ . Since  $k$  is arbitrary, we can treat these as equivalent forms).

We can instead prove the contrapositive: that any integer  $m > 3$  of the forms  $6k, 6k + 2, 6k + 3, 6k + 4$  must be composite. We note that  $6k, 6k + 2, 6k + 4$  can each be written as  $2k$  and  $6k + 3$  is as  $3k$  for an appropriate  $k > 0$ . Thus our original claim is true as well.