CS 70 Spring 2019 | Discrete Mathematics and Probability Theory Satish Rao and Babak Ayazifar | DIS 1A

1 Induction

Prove the following using induction:

(a) For all natural numbers $n > 2$, $2^n > 2n+1$.

(b) For all positive integers n, prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

(c) For all positive natural numbers n, $\frac{5}{4} \cdot 8^n + 3^{3n-1}$ is divisible by 19.

Solution:

(a) The inequality is true for $n = 3$ because $8 > 7$. Let the inequality be true for $n = m$, such that $2^m > 2m+1$. Then,

$$2^{m+1} = 2 \cdot 2^m > 2 \cdot (2m+1) = 4m+2.$$

Considering that $2m > 1$ because m is a positive integer, notice we have:

$$4m+2 = 2m+2m+2 > 2m+1+2 = 2m+3 = 2(m+1)+1$$

We've shown that $2^{m+1} > 2(m+1)+1$, which completes the inductive step.

(b) We can verify that the statement is true for $n = 1$. Assume the statement holds for $n = k$, so that

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}.$$

Then we can write

$$\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= (k+1)\left(\frac{k(2k+1)}{6} + (k+1)\right) \\
&= (k+1)\left(\frac{2k^2+k+6k+6}{6}\right) \\
&= (k+1)\left(\frac{2k^2+7k+6}{6}\right) \\
&= (k+1)\left(\frac{(2k+3)(k+2)}{6}\right) \\
&= \frac{(k+1)(2(k+1)+1)((k+1)+1)}{6},
\end{aligned}$$

as desired. Since we've shown that the statement holds for $n = k+1$, our proof is complete.

(c) For $n = 1$, the statement is "$10+9$ is divisible by 19", which is true. Assume that the statement holds for $n = m$, such that $\frac{5}{4} \cdot 8^m + 3^{3m-1}$ is divisible by 19. Then,

$$\begin{aligned}
\frac{5}{4} \cdot 8^{m+1} + 3^{3(m+1)-1} &= \frac{5}{4} \cdot 8 \cdot 8^m + 3^{3m+2} \\
&= 8 \cdot \frac{5}{4} \cdot 8^m + 3^3 \cdot 3^{3m-1} \\
&= 8 \cdot \frac{5}{4} \cdot 8^m + 8 \cdot 3^{3m-1} + 19 \cdot 3^{3m-1} \\
&= 8\left(\frac{5}{4} \cdot 8^m + 3^{3m-1}\right) + 19 \cdot 3^{3m-1}
\end{aligned}$$

The first term is divisible by the inductive hypothesis, and the second term is clearly divisible by 19. This completes our proof, as we've shown the statement holds for $m+1$.

2 Make It Stronger

Suppose that the sequence $a_1, a_2, \ldots$ is defined by $a_1 = 1$ and $a_{n+1} = 3a_n^2$ for $n \geq 1$. We want to prove that

$$a_n \leq 3^{2^n}$$

for every positive integer n.

(a) Suppose that we want to prove this statement using induction, can we let our induction hypothesis be simply $a_n \leq 3^{2^n}$? Show why this does not work.

(b) Try to instead prove the statement $a_n \leq 3^{2^n - 1}$ using induction. Does this statement imply what you tried to prove in the previous part?

Solution:

(a) Try to prove that for every $n \geq 1$, we have $a_n \leq 3^{2^n}$ by induction.

<u>Base Case</u>: For $n = 1$ we have $a_1 = 1 \leq 3^{2^1} = 9$.
<u>Inductive Hypothesis</u>: For $n \geq 1$ we assume $a_n \leq 3^{2^n}$.
<u>Inductive Step</u>: Assuming the statement is true for an n, we have

$$a_{n+1} = 3a_n^2 \leq 3(3^{2^n})^2 = 3 \times 3^{2 \times 2^n} = 3 \times 3^{2^{n+1}} = 3^{2^{n+1}+1}.$$

However, what we wanted was to get an inequality of the form: $a_{n+1} \leq 3^{2^{n+1}}$. There is an extra $+1$ in the exponent of what we derived.

(b) This time the induction works.

Base Case: For $n = 1$ we have $a_1 = 1 \leq 3^{2-1} = 3$.
Inductive Hypothesis: For $n \geq 1$ we assume $a_n \leq 3^{2^n - 1}$.
Inductive Step: Assuming the hypothesis holds for n, we get

$$a_{n+1} = 3a_n^2 \leq 3 \times (3^{2^n-1})^2 = 3 \times 3^{2\times(2^n-1)} = 3 \times 3^{2^{n+1}-2} = 3^{2^{n+1}-1}.$$

This is exactly the induction hypothesis for $n+1$. Note that for every $n \geq 1$, we have $2^n - 1 \leq 2^n$ and therefore $3^{2^n-1} \leq 3^{2^n}$. This means that our modified hypothesis which we proved here does indeed imply what we wanted to prove in the previous part. This is called "strengthening" the induction hypothesis because we proved a stronger statement and by proving that statement to be true, we proved our original statement to be true as well.

3 Bit String

Prove that every positive integer n can be written with a string of 0s and 1s. In other words, prove that we can write

$$n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + c_0 \cdot 2^0,$$

where $k \in \mathbb{N}$ and $c_k \in \{0, 1\}$.

Solution:

Prove by strong induction on n. Note that this is the first time students will have seen strong induction, so it is important that this problem be done in an interactive way that shows them how simple induction gets stuck.

- Base Case: $n = 1$ can be written with 1×2^0.
- Inductive Hypothesis: Assume that the statement is true for all $1 \leq k \leq n$.
- Inductive Step: If $n+1$ is divisible by 2, then we can apply our inductive hypothesis to $(n+1)/2$ and use its representation to express $n+1$ in the desired form.

$$(n+1)/2 = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + c_0 \cdot 2^0$$
$$n+1 = 2 \cdot (n+1)/2 = c_k \cdot 2^{k+1} + c_{k-1} \cdot 2^k + \cdots + c_1 \cdot 2^2 + c_0 \cdot 2^1.$$

Otherwise, n must be divisible by 2 and have $c_0 = 0$. We can obtain the representation of $n+1$ from n.

$$n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + 0 \cdot 2^0$$
$$n+1 = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + 1 \cdot 2^0$$

Therefore, the statement is true.

Note: In proofs using simple induction, we only use $P(n)$ in order to prove $P(n+1)$. Simple induction gets stuck here because in order to prove $P(n+1)$ in the inductive step, we need to assume more than just $P(n)$. This is because it is not immediately clear how to get a representation for $P(n+1)$ using just $P(n)$, particularly in the case that $n+1$ is divisible by 2. As a result, we assume the statement to be true for all of $1, 2, \ldots, n$ in order to prove it for $P(n+1)$.