

1 Set Operations

- \mathbb{R} , the set of real numbers
- \mathbb{Q} , the set of rational numbers: $\{a/b : a, b \in \mathbb{Z} \wedge b \neq 0\}$
- \mathbb{Z} , the set of integers: $\{\dots, -2, -1, 0, 1, 2, \dots\}$
- \mathbb{N} , the set of natural numbers: $\{0, 1, 2, 3, \dots\}$

- (a) Given a set $A = \{1, 2, 3, 4\}$, what is $\mathcal{P}(A)$ (Power Set)?
- (b) Given a generic set B , how do you describe $\mathcal{P}(B)$ using set comprehension notation? (Set Comprehension is $\{x \mid x \in A\}$.)
- (c) What is $\mathbb{R} \cap \mathcal{P}(A)$?
- (d) What is $\mathbb{R} \cap \mathbb{Z}$?
- (e) What is $\mathbb{N} \cup \mathbb{Q}$?
- (f) What kind of numbers are in $\mathbb{R} \setminus \mathbb{Q}$?
- (g) If $S \subseteq T$, what is $S \setminus T$?

Solution:

(a)

$$\mathcal{P}(A) = \{\{\}, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$$

(b) $\mathcal{P}(B) = \{T \mid T \subseteq B\}$

(c) $\{\}$ or \emptyset

(d) \mathbb{Z}

(e) \mathbb{Q}

(f) The set of irrational numbers

(g) \emptyset

2 Bijections

Consider the function

$$f(x) = \begin{cases} x, & \text{if } x \geq 1; \\ x^2, & \text{if } -1 \leq x < 1; \\ 2x + 3, & \text{if } x < -1. \end{cases}$$

- (a) If the domain and range of f are \mathbb{N} , is f injective (one-to-one), surjective (onto), bijective?
- (b) If the domain and range of f are \mathbb{Z} , is f injective (one-to-one), surjective (onto), bijective?
- (c) If the domain and range of f are \mathbb{R} , is f injective (one-to-one), surjective (onto), bijective?

Solution:

- (a) Yes, Yes, Yes: On \mathbb{N} , f is simply the identity function $id(x) = x$.
- (b) No, No, No: Both -1 and 1 get mapped to 1 (hence not injective) and there is no $x \in \mathbb{Z}$ that gets mapped to -2 (hence not surjective).
- (c) No, Yes, No: -1 and 1 still get mapped to 1 (hence not injective), but every value can be attained (since f is a continuous function and $\lim_{x \rightarrow \pm\infty} f(x) = \pm\infty$), so f is surjective.

3 Unions and Intersections

For each of the following, decide if the expression is "Always Countable", "Always Uncountable", "Sometimes Countable, Sometimes Uncountable".

For the "Always" cases, prove your claim. For the "Sometimes" case, provide two examples – one where the expression is countable, and one where the expression is uncountable.

- (a) $A \cap B$, where A is countable, and B is uncountable
- (b) $A \cup B$, where A is countable, and B is uncountable
- (c) $\bigcap_{i \in A} S_i$ where A is a countable set of indices and each S_i is an uncountable set.

Solution:

- (a) Always countable. $A \cap B$ is a subset of A , which is countable.
- (b) Always uncountable. $A \cup B$ is a superset of B , which is uncountable.

(c) Sometimes countable, sometimes uncountable.

Countable: When the S_i are disjoint, the intersection is empty, and thus countable. For example, let $A = \mathbb{N}$, let $S_i = \{i\} \times \mathbb{R} = \{(i, x) \mid x \in \mathbb{R}\}$. Then, $\bigcap_{i \in A} S_i = \emptyset$.

Uncountable: When the S_i are identical, the intersection is uncountable. Let $A = \mathbb{N}$, let $S_i = \mathbb{R}$ for all i . $\bigcap_{i \in A} S_i = \mathbb{R}$ is uncountable.