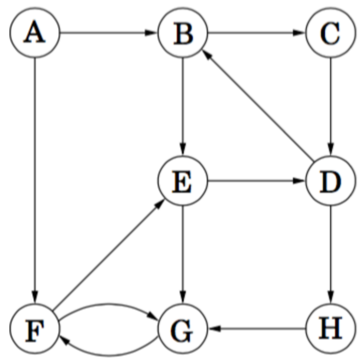
1 Graph Basics

In the first few parts, you will be answering questions on the following graph G.



(a) What are the vertex and edge sets V and E for graph G?

(b) Which vertex has the highest in-degree? Which vertex has the lowest in-degree? Which vertices have the same in-degree and out-degree?

(c) What are the paths from vertex B to F, assuming no vertex is visited twice? Which one is the shortest path?

(d) Which of the following are cycles in G?

i. $(B,C),(C,D),(D,B)$

ii. $(F,G),(G,F)$

iii. $(A,B),(B,C),(C,D),(D,B)$

iv. $(B,C),(C,D),(D,H),(H,G),(G,F),(F,E),(E,D),(D,B)$

(e) Which of the following are walks in G?

i. (E,G)

ii. $(E,G),(G,F)$

iii. $(F,G),(G,F)$

iv. $(A,B),(B,C),(C,D),(H,G)$

v. $(E,G),(G,F),(F,G),(G,C)$

vi. $(E,D),(D,B),(B,E),(E,D),(D,H),(H,G),(G,F)$

(f) Which of the following are tours in G?

i. (E,G)
ii. $(E,G),(G,F)$
iii. $(F,G),(G,F)$
iv. $(E,D),(D,B),(B,E),(E,D),(D,H),(H,G),(G,F)$

In the following three parts, let's consider a general undirected graph G with n vertices ($n \geq 3$).

(g) True/False: If each vertex of G has degree at most 1, then G does not have a cycle.

(h) True/False: If each vertex of G has degree at least 2, then G has a cycle.

(i) True/False: If each vertex of G has degree at most 2, then G is not connected.

Solution:

(a) A graph is specified as an ordered pair $G = (V,E)$, where V is the vertex set and E is the edge set.

$V = \{A,B,C,D,E,F,G,H\}$,
$E = \{(A,B),(A,F),(B,C),(B,E),(C,D),(D,B),(D,H),(E,D),(E,G),(F,E),(F,G),(G,F),$
$(H,G)\}$.

(b) G has the highest in-degree (3). A has the lowest in-degree (0).

$\{B,C,D,E,F,H\}$ all have the same in-degree and out-degree. H and C has in-degree (out-degree) equal to 1 and the other four have in-degree (out-degree) equal to 2.

(c) There are three paths:
$(B,C),(C,D),(D,H),(H,G),(G,F)$
$(B,E),(E,D),(D,H),(H,G),(G,F)$
$(B,E),(E,G),(G,F)$
The first two have length 5, while the last one has length 3, so the last one is the shortest path.

(d) A cycle is a path that starts and ends at the same point. This means that (iii) is not a cycle, since it starts at A but ends at B. In addition, all the vertices $\{v_1,\ldots,v_n\}$ in the cycle $(v_1,v_2),(v_2,v_3),\ldots,(v_{n-1},v_n)$ should be distinct, so (iv) is not a cycle. The correct answers are (i) and (ii).

(e) A walk consists of any sequence of edges such that the endpoint of each edge is the same as the starting vertex of the next edge in the sequence. Example (iv) does not fit this definition–even though it uses only valid edges, the endpoint of the second to last edge in D, while the start point of the next edge is H. Example (v) also is not a walk, since it tries to walk from G to C as its last step, but there is no such edge. All the rest are walks.

(f) A tour is simply a walk that has the same start and end vertex. Only (iii) satisfies this definition. Note in part (d), we already said that (iii) was a cycle–and indeed, all cycles are also tours.

(g) True. In order for there to be a cycle in G starting and ending at some vertex v, we would need at least two edges incident to v: one to leave v at the start of the cycle, and one to return to v at the end. If every vertex has degree at most 1, no vertex has two or more edges incident on it, so no vertex is capable of acting as the start and end point of a cycle.

(h) True. Consider starting a walk at some vertex v_0, and at each step, walking along a previously untraversed edge, stopping when we first visit some vertex w for the second time. If this process terminates, the part of our walk from the first time we visited w until the second time is a cycle. Thus, it remains only to argue this process always terminates.

Each time we take a step from some vertex v, since we are not stopping, we must have visited that vertex exactly once and not yet left. It follows that we have used at most one edge incident with v (either we started at v, or we took an edge into v). Since v has degree at least 2, there must be another edge leaving v for us to take.

(i) False. For example, a 3-cycle (triangle) is connected and every vertex has degree 2.

2 Planarity

Consider graphs with the property T: For every three distinct vertices v_1, v_2, v_3 of graph G, there are at least two edges among them. Prove that if G is a graph on ≥ 7 vertices, and G has property T, then G is nonplanar.

Solution:

Assume G is planar. Take 5 vertices, they cannot form K_5, so some pair v_1, v_2 have no edge between them. The remaining five vertices of G cannot form K_5 either, so there is a second pair v_3, v_4 that have no edge between them. Now consider v_1, v_2 and any other three vertices v_5, v_6, v_7. Since $v_1 v_2$ is not an edge, by property T it must be that $v_1 v$ and $v_2 v$ where $v \in \{v_3, v_4, v_5, v_6, v_7\}$ are edges. Similarly for $v_3, v_4, v_3 v$ and $v_4 v$ where $v \in \{v_1, v_2, v_5, v_6, v_7\}$ are edges. So now any three vertices in $\{v_1, v_2, v_3, v_4\}$ on one side and $\{v_5, v_6, v_7\}$ on the other form an instance of $K_{3,3}$. Contradiction.

The above shows that any graph with 7 vertices and property T is non-planar. Any graph with > 7 vertices and property T will also be non-planar because it will contain a subgraph with 7 vertices and property T.

3 Bipartite Graph

A bipartite graph consists of 2 disjoint sets of vertices (say L and R), such that no 2 vertices in the same set have an edge between them. For example, here is a bipartite graph (with $L = \{$green vertices$\}$ and $R = \{$red vertices$\}$), and a non-bipartite graph.

Prove that a graph is bipartite if and only if it has no tours of odd length.

Solution:

Begin by proving the forward direction: an undirected bipartite graph has no tours of odd length.

Suppose there is a tour in the bipartite graph. Let us start traveling the tour from a node n_0 in L. Since each edge in the graph connects a vertex in L to one in R, the 1st edge in the tour connects our start node n_0 to a node n_1 in R. The 2nd edge in the tour must connect n_1 to a node n_2 in L. Continuing on, the $(2k+1)$-th edge connects node n_{2k} in L to node n_{2k+1} in R, and the $2k$-th edge connects node n_{2k-1} in R to node n_{2k} in L. Since only even numbered edges connect to vertices in L, and we started our tour in L, the tour must end with an even number of edges.

Prove the reverse direction: A undirected graph with no tours of odd length is bipartite.

Take some vertex v. Add all vertices where the shortest path to v is odd, to R. Add all vertices where the shortest path to v is even, to L. If any of the vertices in $u_1, u_2 \in R$ are adjacent, then we have a tour of odd length formed by appending: the shortest path between v and u_1 (odd), the edge (u_1, u_2) (odd), and the shortest path between u_2 and v (odd). This means no two vertices in R are adjacent. Similarly, if any two vertices $v_1, v_2 \in L$ are adjacent, we get a tour of odd length by appending: the shortest path between v and v_1 (even), the edge (v_1, v_2) (odd), and the shortest path between v_2 and v (even). This means no two vertices in L are adjacent either. If there are other connected componenets, we can proceed by choosing a new vertex in each component and repeating this process. Then we will have disjoint L, R which include all vertices.

Figure 1: A bipartite graph (left) and a non-bipartite graph (right).