1 Head Count

Consider a coin with $\mathbb{P}(\text{Heads}) = 2/5$. Suppose you flip the coin 20 times, and define X to be the number of heads.

(a) Name the distribution of X and what its parameters are.

(b) What is $\mathbb{P}(X = 7)$?

(c) What is $\mathbb{P}(X \geq 1)$? Hint: You should be able to do this without a summation.

(d) What is $\mathbb{P}(12 \leq X \leq 14)$?

Solution:

(a) Since we have 20 independent trials, with each trial having a probability $2/5$ of success, $X \sim \text{Binomial}(20, 2/5)$.

(b)

$$\mathbb{P}(X = 7) = \binom{20}{7}\left(\frac{2}{5}\right)^7\left(\frac{3}{5}\right)^{13}.$$

(c)

$$\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X = 0) = 1 - \left(\frac{3}{5}\right)^{20}.$$

(d)

$$\begin{aligned}\mathbb{P}(12 \leq X \leq 14) &= \mathbb{P}(X = 12) + \mathbb{P}(X = 13) + \mathbb{P}(X = 14) \\ &= \binom{20}{12}\left(\frac{2}{5}\right)^{12}\left(\frac{3}{5}\right)^{8} + \binom{20}{13}\left(\frac{2}{5}\right)^{13}\left(\frac{3}{5}\right)^{7} + \binom{20}{14}\left(\frac{2}{5}\right)^{14}\left(\frac{3}{5}\right)^{6}.\end{aligned}$$

2 How Many Queens?

You shuffle a standard 52-card deck, before drawing the first three cards from the top of the pile. Let X denote the number of queens you draw.

(a) What is $\mathbb{P}(X = 0)$, $\mathbb{P}(X = 1)$, $\mathbb{P}(X = 2)$ and $\mathbb{P}(X = 3)$?

(b) What do your answers you computed in part a add up to?

(c) Compute $\mathbb{E}(X)$ from the definition of expectation.

(d) Let X_i be an indicator random variable that equals 1 if the ith card a is queen and 0 otherwise. Are the X_i indicators independent?

Solution:

(a) Calculate each case of X = 0, 1, 2, 3:

We must draw 3 non-queen cards in a row, so the probability is

$$\mathbb{P}(X=0) = \frac{48}{52}\cdot\frac{47}{51}\cdot\frac{46}{50} = \frac{4324}{5525}.$$

Alternatively, every 3-card hand is equally likely, so we can use counting. There are $\binom{52}{3}$ total 3-card hands, and $\binom{48}{3}$ hands with only non-queen cards, which gives us the same result.

$$\mathbb{P}(X=0) = \frac{\binom{48}{3}}{\binom{52}{3}} = \frac{4324}{5525}$$

- We will continue to use counting. The number of hands with exactly one queen amounts to the number of ways to choose 1 queen out of 4, and 2 non-queens out of 48.

$$\mathbb{P}(X=1) = \frac{\binom{4}{1}\binom{48}{2}}{\binom{52}{3}} = \frac{1128}{5525}$$

- Choose 2 queens out of 4, and 1 non-queen out of 48.

$$\mathbb{P}(X=2) = \frac{\binom{4}{2}\binom{48}{1}}{\binom{52}{3}} = \frac{72}{5525}$$

- Choose 3 queens out of 4.

$$\mathbb{P}(X=3) = \frac{\binom{4}{3}}{\binom{52}{3}} = \frac{1}{5525}$$

(b) We check:

$$\mathbb{P}(X=0)+\mathbb{P}(X=1)+\mathbb{P}(X=2)+\mathbb{P}(X=3) = \frac{4324+1128+72+1}{5525} = 1$$

(c) From the definition, $\mathbb{E}(X) = \sum_{k=0}^{3} k\mathbb{P}(X=k)$, so

$$\mathbb{E}(X) = 0\cdot\frac{4324}{5525} + 1\cdot\frac{1128}{5525} + 2\cdot\frac{72}{5525} + 3\cdot\frac{1}{5525} = \frac{3}{13}.$$

(d) No, they are not independent. As an example:

$$\mathbb{P}(X_1=1)\mathbb{P}(X_2=1) = \frac{1}{13}\cdot\frac{1}{13} = \frac{1}{169}$$

However,

$$\mathbb{P}(X_1=1,X_2=1) = \mathbb{P}(\text{the first and second cards are both queens}) = \frac{4}{52}\cdot\frac{3}{51} = \frac{1}{221}.$$

3 Linearity

Solve each of the following problems using linearity of expectation. Explain your methods clearly.

(a) In an arcade, you play game A 10 times and game B 20 times. Each time you play game A, you win with probability $1/3$ (independently of the other times), and if you win you get 3 tickets (redeemable for prizes), and if you lose you get 0 tickets. Game B is similar, but you win with probability $1/5$, and if you win you get 4 tickets. What is the expected total number of tickets you receive?

(b) A monkey types at a 26-letter keyboard with one key corresponding to each of the lower-case English letters. Each keystroke is chosen independently and uniformly at random from the 26 possibilities. If the monkey types 1 million letters, what is the expected number of times the sequence "book" appears?

Solution:

(a) Let A_i be the indicator you win the ith time you play game A and B_i be the same for game B. The expected value of A_i and B_i are

$$\begin{aligned}\mathbb{E}[A_i] &= 1\cdot\frac{1}{3}+0\cdot\frac{2}{3}=\frac{1}{3},\\ \mathbb{E}[B_i] &= 1\cdot\frac{1}{5}+0\cdot\frac{4}{5}=\frac{1}{5}.\end{aligned}$$

Let T_A be the random variable for the number of tickets you win in game A, and T_B be the number of tickets you win in game B.

$$\begin{aligned}\mathbb{E}[T_A+T_B] &= 3\,\mathbb{E}[A_1]+\cdots+3\,\mathbb{E}[A_{10}]+4\,\mathbb{E}[B_1]+\cdots+4\,\mathbb{E}[B_{20}]\\ &= 10\left(3\cdot\frac{1}{3}\right)+20\left(4\cdot\frac{1}{5}\right)=26\end{aligned}$$

(b) There are $1{,}000{,}000-4+1=999{,}997$ places where "book" can appear, each with a (non-independent) probability of $1/26^4$ of happening. If A is the random variable that tells how many times "book" appears, and A_i is the indicator variable that is 1 if "book" appears starting at the ith letter, then

$$\begin{aligned}\mathbb{E}[A] &= \mathbb{E}[A_1+\cdots+A_{999{,}997}]\\ &= \mathbb{E}[A_1]+\cdots+\mathbb{E}[A_{999{,}997}]\\ &= \frac{999{,}997}{26^4}\approx 2.19.\end{aligned}$$