# 1 Clothing Argument

(a) There are four categories of clothings (shoes, trousers, shirts, hats) and we have ten distinct items in each category. How many distinct outfits are there if we wear one item of each category?

(b) How many outfits are there if we wanted to wear exactly two categories?

(c) How many ways do we have of hanging four of our ten hats in a row on the wall? (Order matters.)

(d) We can pack four hats for travels. How many different possibilities for packing four hats are there? Can you express this number in terms of your answer to part (c)?

**Solution:**

(a) $10^4$

(b) $\binom{4}{2} \cdot 10^2$

(c) $\binom{10}{4} \cdot 4! = \dfrac{10!}{6!}$

(d) $\binom{10}{4}$ or written as a function of the previous part, $c/4!$.

# 2 Strings

What is the number of strings you can construct given:

(a) $n$ ones, and $m$ zeroes?

(b) $n_1$ A's, $n_2$ B's and $n_3$ C's?

(c) $n_1, n_2, \ldots, n_k$ respectively of $k$ different letters?

**Solution:**

(a) $\binom{n+m}{n}$

(b) $(n_1+n_2+n_3)!/(n_1! \cdot n_2! \cdot n_3!)$

(c) $(n_1+n_2+\cdots+n_k)!/(n_1! \cdot n_2! \cdots n_k!)$.

## 3 Bit String

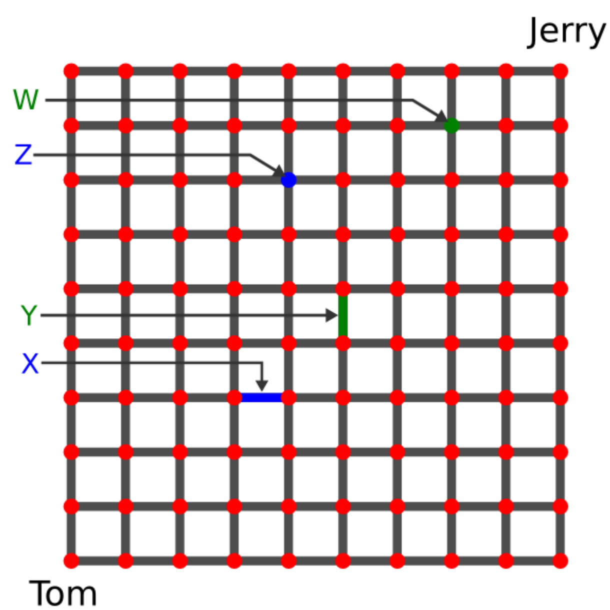
How many bit strings of length 10 contain at least five consecutive 0's?

**Solution:**

One counting strategy is based on where the run of 0's begins. It can begin somewhere between the first digit and the sixth digit, inclusively. If the run begins with the first digit, the first five digits are 0, and there are $2^5 = 32$ choices for the other 5 digits. If the run begins after the first digit, then it must be preceded by a 1. The other four digits can be freely chosen with $2^4 = 16$ possibilities. Thus the total number of 10-bit strings with at least five consecutive 0's is $2^5 + 5 \cdot 2^4 = 112$.

## 4 Maze

Let's assume that Tom is located at the bottom left corner of the $9 \times 9$ maze below, and Jerry is located at the top right corner. Tom of course wants to get to Jerry by the shortest path possible.




(a) How many such shortest paths exist?

(b) How many shortest paths pass through the edge labeled $X$?

(c) The edge labeled $Y$? Both the edges $X$ and $Y$? Neither edge $X$ nor edge $Y$?

(d) How many shortest paths pass through the vertex labeled $Z$? The vertex labeled $W$? Both the vertices $Z$ and $W$? Neither vertex $Z$ nor vertex $W$?

**Solution:**

(a) Each row in the maze has 9 edges, and so does each column. Any shortest path that Tom can take to Jerry will have exactly 9 horizontal edges going right (let's call these "H" edges) and 9 vertical edges going up (let's call these "V" edges).

Observe also that every shortest path from Tom to Jerry can be described by a unique sequence consisting of 9 "H"s and 9 "V"s. For example, one such path is HHHHHHHHHVVVVVVVVV (which represents the path that goes all the way to the right, and then all the way to the top). Conversely, every such sequence of exactly 9 "H"s and 9 "V"s corresponds to a unique shortest path from Tom to Jerry.

Therefore, the number of shortest paths is exactly the same as the number of ways of arranging 9 "H"s and 9 "V"s in a sequence, which is $\binom{18}{9} = 48620$.

(b) For a shortest path to pass through the edge $X$, it has to first get to the left vertex of $X$. So the first portion of the path has to start at the origin, and end at the left vertex of $X$. Using the same logic as above, there are exactly $\binom{6}{3} = 20$ ways to complete this "first leg" of the path (consisting of 3 "H" edges and 3 "V" edges). Having chosen one of these 20 ways, the path has to then go from the right vertex of $X$ to the top right corner of the maze (the "second leg"). This second leg will consist of 5 "H" edges and 6 "V" edges, and using the same logic, there are exactly $\binom{11}{5} = 462$ possibilities. Therefore, the total number of shortest paths that pass through the edge $X$ is $20 \times 462 = 9240$.

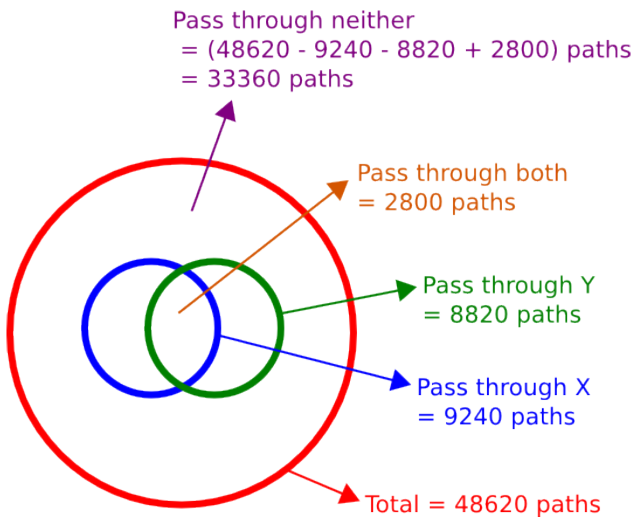
(c) Using similar logic, any shortest path that passes through $Y$ has to consist of 2 legs, the first leg going from the origin to the bottom vertex of $Y$, and the second leg going from the top vertex of $Y$ to the top right corner of the maze. The first leg will consist of exactly 5 "H"s and 4 "V"s, while the second leg will consist of exactly 4 "H"s and 4 "V"s. So the total number of such shortest paths will be $\binom{9}{5} \times \binom{8}{4} = 8820$.

By a similar argument, let's try to figure out how many paths will pass through both $X$ and $Y$. Clearly, any such path has to consist of 3 legs, with the first leg consisting of 3 "H"s and 3 "V"s (going from the origin to the left edge of $X$), the second leg consisting of 1 "H" and 1 "V" (going from the right vertex of $X$ to the bottom vertex of $Y$), and the third leg consisting of 4 "H"s and 4 "V"s (going from the top vertex of $Y$ to the top right corner of the maze). The total number of such shortest paths is therefore $\binom{6}{3} \times \binom{2}{1} \times \binom{8}{4} = 2800$.

Finally, we know that there are 48620 shortest paths in all, of which 9240 pass through $X$, 8820 pass through $Y$, and 2800 pass through both. So the number of paths that pass through neither is 33360 (see the figure above for an intuitive explanation).

(d) This part is very similar in spirit to the previous one, except that in this case, each leg of the path we consider begins exactly where the previous leg ended, and *not* to the right or to the top of where the previous leg ended.

For concreteness, let's find out how many shortest paths pass through vertex $Z$. Observe that for a shortest path to pass through $Z$, it has to first get to $Z$. So the first portion of the path has to

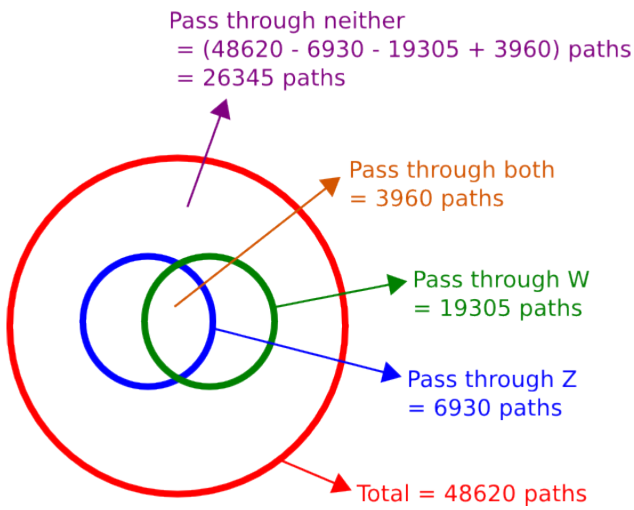



start at the origin, and end at $Z$. Using the same logic as above, there are exactly $\binom{11}{4} = 330$ ways to complete this "first leg" of the path (consisting of 4 "H" edges and 7 "V" edges). Having chosen one of these 330 ways, the path has to then go from $Z$ to the top right corner of the maze. This second leg will consist of 5 "H" edges and 2 "V" edges, and so there are exactly $\binom{7}{2} = 21$ possibilities. Therefore, the total number of shortest paths that pass through the vertex $Z$ is $330 \times 21 = 6930$.

Using similar logic, any shortest path that passes through $W$ has to consist of 2 legs, the first leg going from the origin to $W$, and the second leg going from $W$ to the top right corner of the maze. The first leg will consist of exactly 7 "H"s and 8 "V"s, while the second leg will consist of exactly 2 "H"s and 1 "V". So the total number of such shortest paths will be $\binom{15}{7} \times \binom{3}{1} = 19305$.

By a similar argument, let's try to figure out how many paths will pass through both $Z$ and $W$. Clearly, any such path has to consist of 3 legs, with the first leg consisting of 4 "H"s and 7 "V"s (going from the origin to $Z$), the second leg consisting of 3 "H"s and 1 "V" (going from $Z$ to $W$), and the third leg consisting of 2 "H"s and 1 "V" (going from $W$ to the top right corner of the maze). The total number of such shortest paths is therefore $\binom{11}{4} \times \binom{4}{1} \times \binom{3}{1} = 3960$.

Finally, we know that there are 48620 shortest paths in all, of which 6930 pass through $Z$, 19305 pass through $W$, and 3960 pass through both. So the number of paths that pass through neither is 26345 (see the figure above for an intuitive explanation).


Pass through neither
= (48620 - 6930 - 19305 + 3960) paths
= 26345 paths
Pass through both
= 3960 paths
Pass through W
= 19305 paths
Pass through Z
= 6930 paths
Total = 48620 paths