1 The Count

(a) How many permutations of COSTUME contain "COME" as a substring? How about as a subsequence (meaning the letters of "COME" have to appear in that order, but not necessarily next to each other)?

(b) How many of the first 100 positive integers are divisible by 2, 3, or 5?

(c) How many ways are there to choose five nonnegative integers x_0, x_1, x_2, x_3, x_4 such that $x_0 + x_1 + x_2 + x_3 + x_4 = 100$, and $x_i \equiv i \pmod 5$?

(d) The Count is trying to choose his new 7-digit phone number. Since he is picky about his numbers, he wants it to have the property that the digits are non-increasing when read from left to right. For example, 9973220 is a valid phone number, but 9876545 is not. How many choices for a new phone number does he have?

Solution:

(a) If we need "COME" to be a substring, then this is equivalent to finding the number of ways to arrange 4 items in a row ("COME", "S", "T", and "U"), or $4! = 24$ ways.

If we need "COME" to be a subsequence, let's first choose which 4 positions contain the letters of COME; then, that uniquely determines the positions of C, O, M, E, since they have to be in that order. There are $\binom{7}{4}$ ways to do this, and 3! ways to choose the positions of the remaining three letters, so there are a total of $\binom{7}{4}3! = 210$ valid subsequences.

(b) We use inclusion-exclusion to calculate the number of numbers that satisfy this property. Let A be the set of numbers divisible by 2, B be the set of numbers divisible by 3, and C be the set of numbers divisible by 5. Then, we calculate

$$\begin{aligned}&|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|\\ &\quad=\left\lfloor\frac{100}{2}\right\rfloor+\left\lfloor\frac{100}{3}\right\rfloor+\left\lfloor\frac{100}{5}\right\rfloor-\left\lfloor\frac{100}{6}\right\rfloor-\left\lfloor\frac{100}{10}\right\rfloor-\left\lfloor\frac{100}{15}\right\rfloor+\left\lfloor\frac{100}{30}\right\rfloor\\ &=50+33+20-16-10-6+3=74\end{aligned}$$

numbers.

(c) Let $x_i = 5y_i + i$ for nonnegative integers y_i (we do this because of the modulo conditions). Then, the equation we must satisfy becomes $(5y_0+0)+(5y_1+1)+(5y_2+2)+(5y_3+3)+(5y_4+4) = 100$, which simplifies down to $y_0+y_1+y_2+y_3+y_4 = 18$ for nonnegative integers

y_i. This is a standard stars and bars problem, with 18 objects and 4 dividers. We want to choose how to put the 4 dividers out of the 22 objects, so our answer is $\binom{22}{4}$.

Why is $y_0 + y_1 + y_2 + y_3 + y_4 = 18$ a stars and bars problem? y_i is the number of stars in bin i, and 18 is the number of stars in total. Here are a few possible arrangements:

$$\underbrace{\star\star\star\star\star\star}_{y_0=6} \,|\, \underbrace{\quad}_{y_1=0} \,|\, \underbrace{\star\star\star\star\star\star\star}_{y_2=7} \,|\, \underbrace{\star\star\star\star\star}_{y_3=5} \,|\, \underbrace{\quad}_{y_4=0}$$

$$\underbrace{\star\star}_{y_0=2} \,|\, \underbrace{\star\star\star\star}_{y_1=4} \,|\, \underbrace{\star\star\star\star}_{y_2=4} \,|\, \underbrace{\star\star\star\star\star\star}_{y_3=6} \,|\, \underbrace{\star\star}_{y_4=2}$$

Notice that we arrange the stars into 5 bins using 4 bars. The total number of ways we can arrange 18 stars and 4 bars side-by-side in a line is $\binom{22}{4}$. Why?

- 22 is the total number of objects in the line, 18 stars + 4 bars.
- 4 is the number of bars to be placed in the line.

"22 choose 4", or $\binom{22}{4}$, is the number of ways we can position 4 bars on a line with 18 indistinguishable stars.

(d) This is actually a stars and bars problem in disguise! We have seven positions for digits, and nine dividers to partition these positions into places for nines, places for eights, etc. This is because we know that the digits are non-increasing, so all the nines (if any) must come first, then all the eights (if any), and so on. That means there are a total of 16 objects and dividers, and we are looking for where to put the nine dividers, so our answer is $\binom{16}{9}$.

2 Charming Star

At the end of each day, students will vote for the most charming student. There are 5 candidates and 100 voters. Each voter can only vote once, and all of their votes weigh the same. A "voting combination" is defined by how many votes each candidate receives. In this question, only the number of votes for each candidate matters; it does not matter which specific people voted for each candidate.

(a) How many possible voting combinations are there for the 5 candidates?

(b) How many possible voting combinations are there such that exactly one candidate gets more than 50 votes?

Solution:

(a) Let x_i be the number of votes of the i-th candidates. We would like to find all possible combinations of $(x_1, x_2, x_3, x_4, x_5)$ such that

$$x_1 + x_2 + x_3 + x_4 + x_5 = 100.$$

It is equivalent to selecting $k = 100$ objects from $n = 5$ categories. The number of possible combinations is:

$$\binom{100+5-1}{100} = \binom{104}{100} = 4598126.$$

(b) Now we have a constraint that one of the x_i should be at least 51. Say, let x_1 be at least 51. It is equivalent to giving the first candidate 51 votes at the beginning and then distributing the remaining 49 votes to them again. The number of possible combinations is $\binom{49+5-1}{49}$. Since one of the 5 candidates could have at least 51 votes, the total number of possible voting combinations such that exactly one candidate gets more than 50 votes is:

$$\binom{5}{1}\binom{49+5-1}{49} = 1464125.$$

3 Finicky Bins

If a "finicky" bin has at least 5 balls in it, the 5 balls will fall out and not be counted (e.g., 6 balls in a finicky bin is the same as 1). Suppose we throw 7 indistinguishable balls into 4 finicky bins. How many possible outcomes are there? We consider two outcomes to be the same if they result in the same final distribution of balls in the bins.

Solution:

Consider a normal bin, in which balls do not disappear. With stars and bars we see there are $\binom{10}{3}$ ways to distribute the balls. What if one bin has ≥ 5 balls? There are 4 ways to choose which bin has ≥ 5 balls, and once we throw 5 balls into that bin, we are left to distribute 2 balls among 4 bins in $\binom{5}{3}$ ways.

Now, for the bin in which balls *do* disappear. There are $\binom{10}{3} - 4\binom{5}{3}$ ways to distribute the balls such that no balls disappear. There are $4\binom{5}{3}$ ways to distribute the balls such that 5 balls disappear, except that no matter where the disappearing balls are, there the resulting distribution of balls is the same. Therefore, we have to divide by 4 and we obtain $\binom{5}{3}$ ways to distribute the balls such that 5 balls disappear. In total, we have

$$\binom{10}{3} - 4\binom{5}{3} + \binom{5}{3} = \binom{10}{3} - 3\binom{5}{3}.$$

4 Captain Combinatorial

Please provide combinatorial proofs for the following identities.

(a) $\sum_{i=1}^{n} i\binom{n}{i} = n2^{n-1}$.

(b) $\binom{n}{i} = \binom{n}{n-i}$.

(c) $\sum_{i=1}^{n} i\binom{n}{i}^2 = n\binom{2n-1}{n-1}$.

Solution:

(a) For each i on the LHS, we can think of selecting a team of i members out of a pool of n players, and susequently choosing a captain out of the i team members. The RHS does the same by first choosing the captain out of the n players, and then a subset of the remaining $n-1$ players to constitute the team.

(b) Choosing i players out of n to play on a team is the same as choosing $n-i$ players to not play on the team, i.e. $\binom{n}{i} = \binom{n}{n-i}$.

(c) Assume we have n women and n men. Using part (b) we can rewrite the LHS as $\sum_{i=1}^{n} i\binom{n}{i}\binom{n}{n-i}$, which we can interpret as selecting a team of n players by choosing i women and $n-i$ men, and then choosing one of the women to serve as captain. Again, the RHS first chooses a captain, and then selects a remaining $n-1$ players from all remaining men and women to form the team.