1 Flippin' Coins

Suppose we have a biased coin, with outcomes H and T, with probability of heads $\mathbb{P}[H] = 3/4$ and probability of tails $\mathbb{P}[T] = 1/4$. Suppose we perform an experiment in which we toss the coin 3 times. An outcome of this experiment is (X_1, X_2, X_3), where $X_i \in \{H, T\}$.

(a) What is the *sample space* for our experiment?

(b) Which of the following are examples of *events*? Select all that apply.

- $\{(H,H,T),(H,H),(T)\}$
- $\{(T,H,H),(H,T,H),(H,H,T),(H,H,H)\}$
- $\{(T,T,T)\}$
- $\{(T,T,T),(H,H,H)\}$
- $\{(T,H,T),(H,H,T)\}$

(c) What is the complement of the event $\{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,T,T)\}$?

(d) Let A be the event that our outcome has 0 heads. Let B be the event that our outcome has exactly 2 heads. What is $A \cup B$?

(e) What is the probability of the outcome (H,H,T)?

(f) What is the probability of the event that our outcome has exactly two heads?

Solution:

(a) $\Omega = \{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,H,H),(T,H,T),(T,T,H),(T,T,T)\}$

(b) An event must be a subset of Ω, meaning that it must consist of possible outcomes.

- No
- Yes
- Yes
- Yes
- Yes

(c) $\{(T,H,H),(T,H,T),(T,T,H)\}$

(d) $\{(T,H,H),(H,H,T),(H,T,H),(T,T,T)\}$

(e) $\frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} = \frac{9}{64}$

(f) $\omega \in \{(H,H,T),(H,T,H),(T,H,H)\}$. The probability = $3 \cdot \frac{9}{64} = \frac{27}{64}$.

2 Probability Warm-Up

(a) Suppose that we have a bucket of 30 red balls and 70 blue balls. If we pick 20 balls out of the bucket, what is the probability of getting exactly k red balls (assuming $0 \le k \le 20$) if the sampling is done with replacement?

(b) Same as part (a), but the sampling is without replacement.

(c) If we roll a regular, 6-sided die 5 times. What is the probability that at least one value is observed more than once?

Solution:

(a) Since there is replacement, each time we sample, the probability of choosing a red ball is $30/100$. We repeat this sampling independently 20 times. So

$$\mathbb{P}(k \text{ red balls}) = \binom{20}{k}(0.3)^k(0.7)^{20-k}.$$

(b) Let A be the event of getting exactly k red balls. We note that the size of the sample space is $\binom{100}{20}$, since we are choosing 20 balls out of a total of 100. To find $|A|$, we need to be able to find out how many ways we can choose k red balls and $20-k$ blue balls. So we have that $|A| = \binom{30}{k}\binom{70}{20-k}$. So

$$\mathbb{P}(A) = \frac{\binom{30}{k}\binom{70}{20-k}}{\binom{100}{20}}.$$

(c) Let B be the event that at least one value is observed more than once. We see that $\mathbb{P}(B) = 1 - \mathbb{P}(\overline{B})$. So we need to find out the probability that the values of the 5 rolls are distinct. We see that $\mathbb{P}(\overline{B})$ simply the number of ways to choose 5 numbers (order matters) divided by the sample space (which is 6^5). So

$$\mathbb{P}(\overline{B}) = \frac{6!}{6^5} = \frac{5!}{6^4}.$$

So,

$$\mathbb{P}(B) = 1 - \frac{5!}{6^4}.$$

3 Probability Practice

(a) If we put 5 math, 6 biology, 8 engineering, and 3 physics books on a bookshelf at random, what is the probability that all the math books are together?

(b) A message source M of a digital communication system outputs a word of length 8 characters, with the characters drawn from the ternary alphabet $\{0,1,2\}$, and all such words are equally probable. What is the probability that M produces a word that looks like a byte (*i.e.*, no appearance of '2')?

(c) If five numbers are selected at random from the set $\{1,2,3,\dots,20\}$, what is the probability that their minimum is larger than 5? (A number can be chosen more than once, and the order in which you select the numbers matters)

Solution:

(a) $18!5!/22! = 1/1463$. The 18! comes from 18 "units": 3 physics books, 8 engineering books, 6 biology books and 1 block of math books. The 5! comes from number of ways to arrange the 5 math books within the same block. 22! is just the total number of ways to arrange the books.

(b) $\left(\frac{2^8}{3^8}\right) = 256/6561$. There are 2^8 possible binary-like strings out of 3^8 total possible ternary strings.

(c) $\left(\frac{15^5}{20^5}\right) = 243/1024$. There are 20^5 total possible sequences of numbers we might select. In order for the minimum to be larger than 5, we need to have our entire sequence made up only of the numbers $\{6,7,...,20\}$. Since there are 15 of these numbers, there are 15^5 possible sequences of them.