1 Let's Talk Probability

(a) When is $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ true? What is the general rule that always holds?

(b) When is $\mathbb{P}(A \cap B) = \mathbb{P}(A) * \mathbb{P}(B)$ true? What is the general rule that always holds?

(c) If A and B are disjoint, does that imply they're independent?

Solution:

(a) In general, we know $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$. This is the Inclusion-Exclusion Principle. Therefore if A and B are disjoint, such that $\mathbb{P}(A \cap B) = 0$, then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ holds.

(b) $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ holds if and only if A and B are independent (by definition). The general rule that always holds is $\mathbb{P}(A \cap B) = \mathbb{P}(B|A)\mathbb{P}(A) = \mathbb{P}(A|B)\mathbb{P}(B)$.

(c) No, if two events are disjoint, we cannot conclude they are independent. Consider a roll of a fair six-sided die. Let A be the event that we roll a 1, and let B be the event that we roll a 2. Certainly A and B are disjoint, as $\mathbb{P}(A \cap B) = 0$. But these events are not independent: $\mathbb{P}(B \mid A) = 0$, but $\mathbb{P}(B) = 1/6$.

Since disjoint events have $\mathbb{P}(A \cap B) = 0$, we can see that the only time when A and B are independent is when either $\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$.

2 Aces

Consider a standard 52-card deck of cards:

(a) Find the probability of getting an ace or a red card, when drawing a single card.

(b) Find the probability of getting an ace or a spade, but not both, when drawing a single card.

(c) Find the probability of getting the ace of diamonds when drawing a 5 card hand.

(d) Find the probability of getting exactly 2 aces when drawing a 5 card hand.

(e) Find the probability of getting at least 1 ace when drawing a 5 card hand.

(f) Find the probability of getting at least 1 ace or at least 1 heart when drawing a 5 card hand.

Solution:

(a) Inclusion-Exclusion Principle: $\frac{4}{52} + \frac{26}{52} - \frac{2}{52} = \frac{28}{52} = \frac{7}{13}$.

(b) Inclusion-Exclusion, but we exclude the intersection: $\frac{4}{52} + \frac{13}{52} - 2 \cdot \frac{1}{52} = \frac{15}{52}$.

(c) Ace of diamonds is fixed, but the other 4 cards in the hand can be any other card: $\frac{\binom{51}{4}}{\binom{52}{5}}$.

(d) Account for the number of ways to draw 2 aces and the number of ways to draw 3 non-aces: $\frac{\binom{4}{2} \cdot \binom{48}{3}}{\binom{52}{5}}$.

(e) Complement to getting no aces: $1 - \frac{\binom{48}{5}}{\binom{52}{5}}$.

(f) Complement to getting no aces and no hearts: $1 - \frac{\binom{36}{5}}{\binom{52}{5}}$. This is because $52 - 13 - 3 = 36$, where 13 is the number of hearts and 3 is the number of non-heart aces.

3 Balls and Bins

Throw n balls into n bins.

(a) What is the probability that the first bin is empty?

(b) What is the probability that the first k bins are empty?

(c) Use the union bound to give an upper bound on the probability that at least k bins are empty.

(d) What is the probability that the second bin is empty given that the first one is empty?

(e) Are the events that "the first bin is empty" and "the first two bins are empty" independent?

(f) Are the events that "the first bin is empty" and "the second bin is empty" independent?

Solution:

(a) The probability that ball i does not land in the first bin is $\frac{n-1}{n}$. The probability that all of the balls do not land in the first bin is $\left(\frac{n-1}{n}\right)^n$.

(b) The probability that ball i does not land in the first k bins is $\frac{n-k}{n}$. The probability that all of the balls do not land in the first k bins is $\left(\frac{n-k}{n}\right)^n$.

(c) We use the union bound. Let A be the event that at least k bins are empty. Notice that there are $m = \binom{n}{k}$ sets of k bins out of the total n bins. Then

$$\mathbb{P}(A) = \mathbb{P}\left(\bigcup_{i=1}^{m} A_i\right) \le \sum_{i=1}^{m} \mathbb{P}(A_i)$$

where A_i is the event that the ith set of k bins is empty. We know the probability of the first k bins being empty from part (b), and this is true for any set of k bins, so

$$\mathbb{P}(A_i) = \left(\frac{n-k}{n}\right)^n.$$

Then,

$$\mathbb{P}(A) \le m \cdot \left(\frac{n-k}{n}\right)^n = \binom{n}{k}\left(\frac{n-k}{n}\right)^n.$$

(d) Using Bayes' Rule:

$$\begin{aligned}
\mathbb{P}[\text{2nd bin empty} \mid \text{1st bin empty}] &= \frac{\mathbb{P}[\text{2nd bin empty} \cap \text{1st bin empty}]}{\mathbb{P}[\text{1st bin empty}]} \\
&= \frac{(n-2)^n/n^n}{(n-1)^n/n^n} \\
&= \left(\frac{n-2}{n-1}\right)^n
\end{aligned} \tag{1}$$

Alternate solution:
We know bin 1 is empty, so each ball that we throw can land in one of the remaining $n-1$ bins. We want the probability that bin 2 is empty, which means that each ball cannot land in bin 2 either, leaving $n-2$ bins. Thus for each ball, the probability that bin 2 is empty given that bin 1 is empty is $(n-2)/(n-1)$. For n total balls, this probability is $[(n-2)/(n-1)]^n$.

(e) They are dependent. Knowing the latter means the former happens with probability 1.

(f) In part (c) we calculated the probability that the second bin is empty given that the first bin is empty: $[(n-2)/(n-1)]^n$. The probability that the second bin is empty (without any prior information) is $[(n-1)/n]^n$. Since these probabilities are not equal, the events are dependent.