CS 70 Discrete Mathematics and Probability Theory
Spring 2019 Babak Ayazifar and Satish Rao **DIS 9A**

1 How Many Queens?

You shuffle a standard 52-card deck, before drawing the first three cards from the top of the pile. Let X denote the number of queens you draw.

(a) What is $\mathbb{P}(X = 0)$?

(b) What is $\mathbb{P}(X = 1)$?

(c) What is $\mathbb{P}(X = 2)$?

(d) What is $\mathbb{P}(X = 3)$?

(e) Do the answers you computed in parts (a) through (d) add up to 1, as expected?

(f) Compute $\mathbb{E}(X)$ from the definition of expectation.

(g) Suppose we define indicators X_i, $1 \le i \le 3$, where X_i is the indicator variable that equals 1 if the ith card is a queen and 0 otherwise. Compute $\mathbb{E}(X)$ using linearity of expectation.

(h) Are the X_i indicators independent? Does this affect your solution to part (g)?

Solution:

(a) We must draw 3 non-queen cards in a row, so the probability is

$$\mathbb{P}(X = 0) = \frac{48}{52} \cdot \frac{47}{51} \cdot \frac{46}{50} = \frac{4324}{5525}.$$

Alternatively, every 3-card hand is equally likely, so we can use counting. There are $\binom{52}{3}$ total 3-card hands, and $\binom{48}{3}$ hands with only non-queen cards, which gives us the same result.

$$\mathbb{P}(X = 0) = \frac{\binom{48}{3}}{\binom{52}{3}} = \frac{4324}{5525}$$

(b) We will continue to use counting. The number of hands with exactly one queen amounts to the number of ways to choose 1 queen out of 4, and 2 non-queens out of 48.

$$\mathbb{P}(X = 1) = \frac{\binom{4}{1}\binom{48}{2}}{\binom{52}{3}} = \frac{1128}{5525}$$

(c) Choose 2 queens out of 4, and 1 non-queen out of 48.

$$\mathbb{P}(X=2) = \frac{\binom{4}{2}\binom{48}{1}}{\binom{52}{3}} = \frac{72}{5525}$$

(d) Choose 3 queens out of 4.

$$\mathbb{P}(X=3) = \frac{\binom{4}{3}}{\binom{52}{3}} = \frac{1}{5525}$$

(e) We check:

$$\mathbb{P}(X=0)+\mathbb{P}(X=1)+\mathbb{P}(X=2)+\mathbb{P}(X=3) = \frac{4324+1128+72+1}{5525} = 1$$

(f) From the definition, $\mathbb{E}(X) = \sum_{k=0}^{3} k\mathbb{P}(X=k)$, so

$$\mathbb{E}(X) = 0\cdot\frac{4324}{5525} + 1\cdot\frac{1128}{5525} + 2\cdot\frac{72}{5525} + 3\cdot\frac{1}{5525} = \frac{3}{13}.$$

(g) We know that $\mathbb{E}(X_i) = \mathbb{P}(\text{card } i \text{ is a queen}) + 0\cdot\mathbb{P}(\text{card } i \text{ is not a queen}) = 1/13$, so

$$\mathbb{E}(X) = \mathbb{E}(X_1)+\mathbb{E}(X_2)+\mathbb{E}(X_3) = \frac{1}{13}+\frac{1}{13}+\frac{1}{13} = \frac{3}{13}.$$

Notice how much faster it was to compute the expectation using indicators!

(h) No, they are not independent. As an example:

$$\mathbb{P}(X_1=1)\mathbb{P}(X_2=1) = \frac{1}{13}\cdot\frac{1}{13} = \frac{1}{169}$$

However,

$$\mathbb{P}(X_1=1, X_2=1) = \mathbb{P}(\text{the first and second cards are both queens}) = \frac{4}{52}\cdot\frac{3}{51} = \frac{1}{221}.$$

Even though the indicators are not independent, this does not change our answer for part (g). Linearity of expectation *always* holds, which makes it an extremely powerful tool.

2 More Aces in a Deck

There are four aces in a deck. Suppose you shuffle the deck; define the random variables:

$$\begin{aligned}
X_1 &= \text{number of non-ace cards before the first ace}\\
X_2 &= \text{number of non-ace cards between the first and second ace}\\
X_3 &= \text{number of non-ace cards between the second and third ace}\\
X_4 &= \text{number of non-ace cards between the third and fourth ace}\\
X_5 &= \text{number of non-ace cards after the fourth ace}
\end{aligned}$$

1. What is $X_1 + X_2 + X_3 + X_4 + X_5$?
2. Argue that the X_i random variables all have the same distribution. Are they independent?

Solution:

1. $X_1 + X_2 + X_3 + X_4 + X_5$ must sum up to the number of non-ace cards in the deck, which is 48.
2. This is true by symmetry; the aces split the deck into 5 equal sections, and by symmetry, the expected number of cards in each section is equal. No they are not independent. For example, if $X_1 = 48$ then we know that $X_2, X_3, X_4 = 0$.

3 Head Count

Consider a coin with $\mathbb{P}(\text{Heads}) = 2/5$. Suppose you flip the coin 20 times, and define X to be the number of heads.

(a) What is the distribution of X?

(b) What is $\mathbb{P}(X = 7)$?

(c) What is $\mathbb{P}(X \geq 1)$?

(d) What is $\mathbb{P}(12 \leq X \leq 14)$?

Solution:

(a) Since we have 20 independent trials, with each trial having a probability $2/5$ of success, $X \sim \text{Binomial}(20, 2/5)$.

(b)

$$\mathbb{P}(X = 7) = \binom{20}{7}\left(\frac{2}{5}\right)^7\left(\frac{3}{5}\right)^{13}.$$

(c)

$$\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X = 0) = 1 - \left(\frac{3}{5}\right)^{20}.$$

(d)

$$\begin{aligned}\mathbb{P}(12 \leq X \leq 14) &= \mathbb{P}(X = 12) + \mathbb{P}(X = 13) + \mathbb{P}(X = 14) \\ &= \binom{20}{12}\left(\frac{2}{5}\right)^{12}\left(\frac{3}{5}\right)^{8} + \binom{20}{13}\left(\frac{2}{5}\right)^{13}\left(\frac{3}{5}\right)^{7} + \binom{20}{14}\left(\frac{2}{5}\right)^{14}\left(\frac{3}{5}\right)^{6}.\end{aligned}$$