1 Numbered Balls

Suppose you have a bag containing seven balls numbered $0, 1, 1, 2, 3, 5, 8$.

(a) You perform the following experiment: pull out a single ball and record its number. What is the expected value of the number that you record?

(b) You repeat the experiment from part (a), except this time you pull out two balls together and record their total. What is the expected value of the total that you record?

Solution:

(a) Let X be the number that you record. Each ball is equally likely to be chosen, so

$$\mathbb{E}[X] = \sum_x x \cdot \mathbb{P}(X = x) = 0 \times \frac{1}{7} + 1 \times \frac{2}{7} + 2 \times \frac{1}{7} + 3 \times \frac{1}{7} + 5 \times \frac{1}{7} + 8 \times \frac{1}{7} = \frac{20}{7}.$$

As demonstrated here, the expected value of a random variable need not, and often is not, a feasible value of that random variable (there is no outcome ω for which $X(\omega) = 20/7$).

(b) Let X_1 be the number on the first ball that you pull out, and X_2 be the number on the second ball that you pull out. Then $X = X_1 + X_2$, and

$$\mathbb{E}[X] = \mathbb{E}[X_1 + X_2] = \mathbb{E}[X_1] + \mathbb{E}[X_2] = \frac{20}{7} + \frac{20}{7} = \frac{40}{7}$$

where the second equality applies linearity of expectation. Note that using linearity of expectation does *not* require X_1 and X_2 to be independent! Indeed, X_1 and X_2 are not independent because $\mathbb{P}(X_1 = 0) = 1/7$ but $\mathbb{P}(X_1 = 0 \mid X_2 = 0) = 0$.

2 Airport Revisited

(a) Suppose that there are n airports arranged in a circle. A plane departs from each airport, and randomly chooses an airport to its left or right to fly to. What is the expected number of empty airports after all planes have landed?

(b) Now suppose that we still have n airports, but instead of being arranged in a circle, they form a general graph, where each airport is denoted by a vertex, and an edge between two airports indicates that a flight is permitted between them. There is a plane departing from each airport and randomly chooses a neighboring destination where a flight is permitted. What is the expected number of empty airports after all planes have landed? (Express your answer in terms of $N(i)$ - the set of neighboring airports of airport i, and $\deg(i)$ - the number of neighboring airports of airport i).

Solution:

(a) Let X_i be the indicator variable denoting whether airport i ends up empty. This can happen if and only if planes from both of its neighboring airports are flying elsewhere, and this happens with a probability of $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$. Hence the expected number of empty airports is

$$\mathbb{E}[X_1 + \ldots + X_n] = \mathbb{E}[X_1] + \ldots + \mathbb{E}[X_n] = \frac{n}{4}$$

where we use linearity of expectation in the first step.

(b) Similar to the previous part, we now have $\mathbb{E}[X_i] = P(X_i = 1) = \prod_{j \in N(i)} \left(1 - \frac{1}{\deg(j)}\right)$. Hence

$$\mathbb{E}[X_1 + \ldots + X_n] = \mathbb{E}[X_1] + \ldots + \mathbb{E}[X_n] = \sum_{i=1}^{n} \prod_{j \in N(i)} \left(1 - \frac{1}{\deg(j)}\right)$$

3 Telebears

Lydia has just started her CalCentral enrollment appointment. She needs to register for a marine science class and CS 70. There are no waitlists, and she can attempt to enroll once per day in either class or both. The CalCentral enrollment system is strange and picky, so the probability of enrolling in the marine science class is μ and the probability of enrolling in CS 70 is κ. These events are independent. Let M be the number of days it takes to enroll in the marine science class, and C be the number of days it takes to enroll in CS 70.

(a) What distribution do M and C follow? Are M and C independent?

(b) For some integer $k \geq 1$, what is $\mathbb{P}[C \geq k]$?

(c) For some integer $k \geq 1$, what is the probability that she is enrolled in both classes before day k?

Solution:

(a) $M \sim \text{Geometric}(\mu)$, $C \sim \text{Geometric}(\kappa)$. Yes they are independent.

(b) We are looking for the probability that it takes at least k days to enroll in CS 70. Using the geometric distribution, this is $(1-\kappa)^{k-1}$.

(c) Use independence. Let X be the number of days before she is enrolled in both.

$$\begin{aligned}\mathbb{P}[X < k] &= \mathbb{P}[M < k]\mathbb{P}[C < k] = (1 - \mathbb{P}[M \geq k])(1 - \mathbb{P}[C \geq k]) \\ &= (1 - (1-\mu)^{k-1})(1 - (1-\kappa)^{k-1})\end{aligned}$$