CS 70 Discrete Mathematics and Probability Theory
Spring 2021 DIS 0A

1 Truth Tables

Determine whether the following equivalences hold, by writing out truth tables. Clearly state whether or not each pair is equivalent.

(a) $P \wedge (Q \vee P) \equiv P \wedge Q$

(b) $(P \vee Q) \wedge R \equiv (P \wedge R) \vee (Q \wedge R)$

(c) $(P \wedge Q) \vee R \equiv (P \vee R) \wedge (Q \vee R)$

Solution:

(a) Not equivalent.

P	Q	$P \wedge (Q \vee P)$	$P \wedge Q$
T	T	T	T
T	F	T	F
F	T	F	F
F	F	F	F

(b) Equivalent.

P	Q	R	$(P \vee Q) \wedge R$	$(P \wedge R) \vee (Q \wedge R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

(c) Equivalent.

P	Q	R	$(P \wedge Q) \vee R$	$(P \vee R) \wedge (Q \vee R)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	F	F
F	F	T	T	T
F	F	F	F	F

2 XOR

The truth table of XOR (denoted by $\oplus$) is as follows.

A	B	A $\oplus$ B
F	F	F
F	T	T
T	F	T
T	T	F

1. Express XOR using only $(\wedge, \vee, \neg)$ and parentheses.
2. Does $(A \oplus B)$ imply $(A \vee B)$? Explain briefly.
3. Does $(A \vee B)$ imply $(A \oplus B)$? Explain briefly.

Solution:

1. These are all correct:
   - $A \oplus B = (A \wedge \neg B) \vee (\neg A \wedge B)$
     Notice that there are only two instances when $A \oplus B$ is true: (1) when A is true and B is false, or (2) when B is true and A is false. The clause $(A \wedge \neg B)$ is only true when (1) is, and the clause $(\neg A \wedge B)$ is only true when (2) is.
   - $A \oplus B = (A \vee B) \wedge (\neg A \vee \neg B)$
     Another way to think about XOR is that exactly one of A and B needs to be true. This also means exactly one of $\neg A$ and $\neg B$ needs to be true. The clause $(A \vee B)$ tells us *at least* one of A and B needs to be true. In order to ensure that one of A or B is also false, we need the clause $(\neg A \vee \neg B)$ to be satisfied as well.
   - $A \oplus B = (A \vee B) \wedge \neg(A \wedge B)$
     This is the same as the previous, with De Morgan's law applied to equate $(\neg A \vee \neg B)$ to $\neg(A \wedge B)$.

2. Yes. $(A \oplus B) \implies (A \wedge \neg B) \vee (\neg A \wedge B) \implies (A \vee B)$. When $(A \oplus B)$ is true, at least one of A or B is true, which makes $(A \vee B)$ true as well.

3. No. When A and B are both true, then $(A \vee B)$ is true, but $(A \oplus B)$ is false.

3 Converse and Contrapositive

Consider the statement "if a natural number is divisible by 4, it is divisible by 2".

(a) Write the statement in propositional logic. Prove that it is true or give a counterexample.

(b) Write the inverse of the implication in English and in propositional logic. Prove that it is true or give a counterexample. (The inverse of an implication $P \implies Q$ is $\neg P \implies \neg Q$.)

(c) Write the converse of the implication in English and in propositional logic. Prove that it is true or give a counterexample.

(d) Write the contrapositive of the implication in English and in propositional logic. Prove that it is true or give a counterexample.

Solution:

(a) $(\forall x \in \mathbb{N})\ (4 \mid x \implies 2 \mid x)$. This statement is true. We know that if x is divisible by 4, we can write x as $4k$ for some integer k. But $4k = (2 \cdot 2)k = 2(2k)$, where $2k$ is also an integer. Thus, x must also be divisible by 2, since it can be written as 2 times an integer.

(b) The inverse is that if a natural number is not divisible by 4, it is not divisible by 2: $(\forall x \in \mathbb{N})\ (4 \nmid x \implies 2 \nmid x)$. This is false, since 2 is not divisible by 4, but is divisible by 2.

(c) The converse is that any natural number that is divisible by 2 is also divisible by 4: $(\forall x \in \mathbb{N})\ (2 \mid x \implies 4 \mid x)$. Again, this is false, since 2 is divisible by 2 but not by 4.

(d) The contrapositive is that any natural number that is not divisible by 2 is not divisible by 4: $(\forall x \in \mathbb{N})\ (2 \nmid x \implies 4 \nmid x)$. To show that this is true, first consider that saying that x is not divisible by 2 is equivalent to saying that $x/2$ is not an integer. And if we divide a non-integer by an integer, we get back another non-integer–so $(x/2)/2 = x/4$ must also not be an integer. But that is exactly the same as saying that x is not divisible by 4.

Note that the inverse and the converse will always be contrapositives of each other, and so will always be logically equivalent.

4 Equivalences with Quantifiers

Evaluate whether the expressions on the left and right sides are equivalent in each part, and briefly justify your answers.

(a)	$\forall x\ \big((\exists y\ Q(x,y)) \Rightarrow P(x)\big)$	$\forall x\ \exists y\ \big(Q(x,y) \Rightarrow P(x)\big)$
(b)	$\neg \exists x\ \forall y\ \big(P(x,y) \Rightarrow \neg Q(x,y)\big)$	$\forall x\ \big((\exists y\ P(x,y)) \land (\exists y\ Q(x,y))\big)$
(c)	$\forall x\ \exists y\ \big(P(x) \Rightarrow Q(x,y)\big)$	$\forall x\ \big(P(x) \Rightarrow (\exists y\ Q(x,y))\big)$

Solution:

(a) Not equivalent.

Justification: We can rewrite the left side as as $\forall x\ \big((\neg(\exists y\ Q(x,y))) \lor P(x)\big)$ and the right side as $\forall x\ \exists y\ \big(\neg Q(x,y) \lor P(x)\big)$ Applying the negation on the left side of the equivalence $(\neg(\exists y Q(x,y)))$ changes the $\exists y$ to $\forall y$, and the two sides are clearly not the same. Another approach to the problem is to consider by linguistic example. Let x and y span the universe of all people, and let $Q(x,y)$ mean "Person x is Person y's offspring", and let $P(x)$ mean "Person x likes tofu". The right side claims that, for all Persons x, there exists some Person y such that either Person x is not Person y's offspring or that Person x likes tofu. The left side claims that, for all Persons x, if there exists a parent of Person x, then Person x likes tofu. Obviously, these are not the same.

(b) Not equivalent.

Justification: Using De Morgan's Law to distribute the negation on the left side yields

$$\forall x\ \exists y\ \big(P(x,y) \land Q(x,y)\big).$$

But $\exists$ does not distribute over $\land$. There could exist different values of y such that $P(x,y)$ and $Q(x,y)$ for a given x, but not necessarily the same value.

(c) Equivalent.

Justification: We can rewrite the left side as $\forall x\ \exists y\ \big(\neg P(x) \lor Q(x,y)\big)$ and the right side as $\forall x\ \big(\neg P(x) \lor (\exists y\ Q(x,y))\big)$. Clearly, the two sides are the same if $\neg P(x)$ is true. If $\neg P(x)$ is false, then the two sides are still the same, because $\forall x\ \exists y\ \big(\text{False} \lor Q(x,y)\big) \equiv \forall x\ \big(\text{False} \lor (\exists y\ Q(x,y))\big)$.