1 DeMorgan's Laws

Use truth tables to show that $\neg(A \vee B) \equiv \neg A \wedge \neg B$ and $\neg(A \wedge B) \equiv \neg A \vee \neg B$. These two equivalences are known as DeMorgan's Laws.

Solution:

A	B	$A \vee B$	$\neg(A \vee B)$	$\neg A \wedge \neg B$
T	T	T	F	F
T	F	T	F	F
F	T	T	F	F
F	F	F	T	T

A	B	$A \wedge B$	$\neg(A \wedge B)$	$\neg A \vee \neg B$
T	T	T	F	F
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

2 XOR

The truth table of XOR (denoted by $\oplus$) is as follows.

A	B	A $\oplus$ B
F	F	F
F	T	T
T	F	T
T	T	F

1. Express XOR using only $(\wedge, \vee, \neg)$ and parentheses.
2. Does $(A \oplus B)$ imply $(A \vee B)$? Explain briefly.
3. Does $(A \vee B)$ imply $(A \oplus B)$? Explain briefly.

Solution:

1. These are all correct:

- $A \oplus B = (A \land \neg B) \lor (\neg A \land B)$
  Notice that there are only two instances when $A \oplus B$ is true: (1) when A is true and B is false, or (2) when B is true and A is false. The clause $(A \land \neg B)$ is only true when (1) is, and the clause $(\neg A \land B)$ is only true when (2) is.
- $A \oplus B = (A \lor B) \land (\neg A \lor \neg B)$
  Another way to think about XOR is that exactly one of A and B needs to be true. This also means exactly one of $\neg A$ and $\neg B$ needs to be true. The clause $(A \lor B)$ tells us *at least* one of A and B needs to be true. In order to ensure that one of A or B is also false, we need the clause $(\neg A \lor \neg B)$ to be satisfied as well.
- $A \oplus B = (A \lor B) \land \neg(A \land B)$
  This is the same as the previous, with De Morgan's law applied to equate $(\neg A \lor \neg B)$ to $\neg(A \land B)$.

2. Yes. $(A \oplus B) \implies (A \land \neg B) \lor (\neg A \land B) \implies (A \lor B)$.

3. No. When A and B are both true, then $(A \lor B)$ is true, but $(A \oplus B)$ is false.

3 Numbers of Friends

Prove that if there are $n \geq 2$ people at a party, then at least 2 of them have the same number of friends at the party. Assume that friendships are always reciprocated: that is, if Alice is friends with Bob, then Bob is also friends with Alice.
(Hint: The Pigeonhole Principle states that if n items are placed in m containers, where $n > m$, at least one container must contain more than one item. You may use this without proof.)

Solution:

We will prove this by contradiction. Suppose the contrary that everyone has a different number of friends at the party. Since the number of friends that each person can have ranges from 0 to $n-1$, we conclude that for every $i \in \{0, 1, \ldots, n-1\}$, there is exactly one person who has exactly i friends at the party. In particular, there is one person who has $n-1$ friends (i.e., friends with everyone), and there is one person who has 0 friends (i.e., friends with no one), which is a contradiction.

Here, we used the pigeonhole principle because assuming for contradiction that everyone has a different number of friends gives rise to n possible containers. Each container denotes the number of friends that a person has, so the containers can be labelled $0, 1, ..., n-1$. The objects assigned to these containers are the people at the party. However, containers 0, $n-1$ or both must be empty since these two containers cannot be occupied at the same time. This means that we are assigning n people to at most $n-1$ containers, and by the pigeonhole principle, at least one of the $n-1$ containers has to have two or more objects i.e. at least two people have to have the same number of friends.

4 Proof Practice

(a) Prove that $\forall n \in \mathbb{N}$, if n is odd, then n^2+1 is even.

(b) Prove that $\forall x, y \in \mathbb{R}$, $\min(x,y) = (x+y-|x-y|)/2$.

(c) Prove that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

(d) Suppose $A \subseteq B$. Prove $\mathscr{P}(A) \subseteq \mathscr{P}(B)$.

Solution:

(a) We will use a direct proof. Assume n is odd. By the definition of odd numbers, $n = 2k+1$ for some natural number k. Substituting into the expression n^2+1, we get $(2k+1)^2+1$. Simplifying the expression yields $4k^2+4k+2$. This can be rewritten as $2 \times (2k^2+2k+1)$. Since $2k^2+2k+1$ is a natural number, by the definition of even numbers, n^2+1 is even.

(b) We will use a proof by cases. We know the following about the absolute value function for real number z.

$$|z| = \begin{cases} z, & z \geq 0 \\ -z, & z < 0 \end{cases}$$

Case 1: $x < y$. This means $|x-y| = y-x$. Substituting this into the formula on the right hand side, we get

$$\frac{x+y-y+x}{2} = x = \min(x,y).$$

Case 2: $x \geq y$. This means $|x-y| = x-y$. Substituting this into the formula on the right hand side, we get

$$\frac{x+y-x+y}{2} = y = \min(x,y).$$

(c)

$$\sum_{i=1}^{n} i = 1+2+\cdots+n$$

$$2\sum_{i=1}^{n} i = (1+n)+(2+(n-1))+\cdots+(n+1) = (n+1)n$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

(d) Suppose $A' \in \mathscr{P}(A)$, that is, $A' \subseteq A$ (by the definition of the power set). We must prove that for any such A', we also have that $A' \in \mathscr{P}(B)$, that is, $A' \subseteq B$.

Let $x \in A'$. Then, since $A' \subseteq A$, $x \in A$. Since $A \subseteq B$, $x \in B$. We have shown $(\forall x \in A')\, x \in B$, so $A' \subseteq B$.

Since the previous argument works for any $A' \subseteq A$, we have proven $(\forall A' \in \mathscr{P}(A))\, A' \in \mathscr{P}(B)$. So, we conclude $\mathscr{P}(A) \subseteq \mathscr{P}(B)$ as desired.