1 Probabilistically Buying Probability Books

Chuck will go shopping for probability books for K hours. Here, K is a random variable and is equally likely to be 1, 2, or 3. The number of books N that he buys is random and depends on how long he shops. We are told that

$$\mathbb{P}[N=n|K=k] = \begin{cases} \frac{c}{k} & \text{for } n=1,\ldots,k \\ 0 & \text{otherwise} \end{cases}$$

for some constant c.

(a) Compute c.

(b) Find the joint distribution of K and N.

(c) Find the marginal distribution of N.

Solution:

(a) For any k, we know that probabilities conditioned on $K=k$ must sum to 1, i.e

$$\sum_n \mathbb{P}[N=n|K=k] = 1 \quad ,$$

so it must be that

$$1 = \sum_{n=1}^{k} \mathbb{P}[N=n|K=k] = k \times \frac{c}{k} = c \quad .$$

Thus, $c=1$.

(b) The joint distribution specifies $\mathbb{P}[N=n \cap K=k]$ for all n and k. Note that

$$\mathbb{P}[N=n \cap K=k] = \mathbb{P}[N=n|K=k]\mathbb{P}[K=k]$$

and we know $\mathbb{P}[N=n|K=k]$ and $\mathbb{P}[K=k]$ (it says all $k \in \{1,2,3\}$ are equally likely). We use this formula to calculate $\mathbb{P}[N=n \cap K=k]$ for each n,k and list the result in a table:

$n \setminus k$	1	2	3
1	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{9}$
2	0	$\frac{1}{6}$	$\frac{1}{9}$
3	0	0	$\frac{1}{9}$

Alternatively, we can define the joint distribution as a formula with specified domain. $\mathbb{P}[N = n, K = k] = \mathbb{P}[N = n \mid K = k]\mathbb{P}[K = k] = \frac{1}{k}\frac{1}{3}$ whenever it is nonzero. So,

$$\mathbb{P}[N = n, K = k] = \begin{cases} \frac{1}{3k} & k \in \{1,2,3\}, n \in \{1,\dots,k\} \\ 0 & \text{otherwise} \end{cases}$$

(c) The marginal distribution of N is given by the value of $\mathbb{P}[N = n]$, for each possible value of n. By the total probability rule,

$$\mathbb{P}[N = n] = \mathbb{P}[N = n \cap K = 1] + \mathbb{P}[N = n \cap K = 2] + \mathbb{P}[N = n \cap K = 3] \ .$$

Thus, we get

$$\mathbb{P}[N = n] = \begin{cases} \frac{1}{3} + \frac{1}{6} + \frac{1}{9} & \text{if } n = 1 \\ \frac{1}{6} + \frac{1}{9} & \text{if } n = 2 \\ \frac{1}{9} & \text{if } n = 3 \end{cases} = \begin{cases} \frac{11}{18} & \text{if } n = 1 \\ \frac{5}{18} & \text{if } n = 2 \\ \frac{2}{18} & \text{if } n = 3 \end{cases}$$

2 Joint Distributions

(a) Give examples of joint distribution over discrete random variables X and Y such that $\mathbb{E}[XY] \neq \mathbb{E}[X]\mathbb{E}[Y]$.

(b) Give examples of joint distribution over discrete random variables X and Y such that $\mathbb{E}[XY] = 0$, $\mathbb{E}[X] = 0$, and $\mathbb{E}[Y] = 0$, but X and Y are not independent.

(c) Suppose that X_i, $i = 1,\dots,n$ are binary-valued random variables. How many parameters are required to parameterize the joint distribution $\mathbb{P}(X_1 = x_1,\dots,X_n = x_n)$?

(d) Continuing from the previous part, suppose that all X_is are independent. How many parameters are required to parameterize the joint distribution?

Solution:

(a) Let $P(X = 1) = \frac{1}{2}$, $P(X = -1) = \frac{1}{2}$, and $Y \equiv X$. Then $\mathbb{E}[XY] = \mathbb{E}[X^2] = 1$ and $\mathbb{E}[X]\mathbb{E}[Y] = 0$.

(b) An example is given by $P(X = -1, Y = \frac{1}{3}) = P(X = 1, Y = \frac{1}{3}) = P(X = 0, Y = -\frac{2}{3}) = \frac{1}{3}$.

(c) There are $2^n - 1$ parameters required. There are 2^n entries in the joint distribution, but the values must sum to 1, so there are only $2^n - 1$ free parameters.

(d) Each random variable requires only 1 parameter, so in total only n parameters are required.

3 Binomial Conditioning

Let $n \in \mathbb{Z}_+$ and $p, q \in [0,1]$. Let $X \sim \text{Binomial}(n, p)$ and suppose that conditioned on $X = x$, $Y \sim \text{Binomial}(x, q)$. What is the unconditional distribution of Y?

Solution:

Y takes on values in $\{0, \ldots, n\}$. What we want to compute is the probability that $Y = y$ for each $y \in \{0, \ldots, n\}$. The way we will do this is by going to the joint distribution $\mathbb{P}(X = x, Y = y)$, pick out the entries that satisfy $Y = y$, and then add them all together. Note that in order for $\mathbb{P}(X = x, Y = y) > 0$, we need $y \le x$. To get the joint distribution, we have

$$\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x)\mathbb{P}(Y = y \mid X = x) = \binom{n}{x} p^x (1-p)^{n-x} \binom{x}{y} q^y (1-q)^{x-y}$$

Then, iterating through all the possible values of x while keeping y fixed yields the sum

$$\mathbb{P}(Y = y) = \sum_{x=y}^{n} \binom{n}{x} p^x (1-p)^{n-x} \binom{x}{y} q^y (1-q)^{x-y}$$

It is important to understand how we carried out the algebra here, because it is a general technique/methodology to answer these kinds of questions: get access to the joint distribution and sum together all elements in that joint distribution that satisfy the condition.

Finding this sum is a little involved, and we defer this discussion to the end of the solutions. However, we might step back a little and revisit the Binomial process to see if we can arrive at the answer. We can interpret the problem as follows. Let's say there are n people lined up. Each of them will flip a coin with bias p. If they flip tails, they are out of the game, and have to sit down. Otherwise, they stay in the game, and now flip a coin with bias q. If that coin ends up tails, they have to sit down. X represents the number of people that survived the first round, and Y represents the number of people that survived the second round.

We can get access to Y directly when we observe how one person can make it to the end. They must flip heads on the first round, then heads on the second round, and the probability this happens is pq. Each person makes it to the end independently, so therefore, the number of people at the end must be binomially distributed with probability pq.

Therefore, we see that $Y \sim \text{Binomial}(n, pq)$.

Now, let's see how we could have carried out the algebra. Recall that we were trying to evaluate this sum:

$$\mathbb{P}(Y = y) = \sum_{x=y}^{n} \binom{n}{x} p^x (1-p)^{n-x} \binom{x}{y} q^y (1-q)^{x-y}$$

Let's first break apart the binomial coefficients:

$$\sum_{x=y}^{n} \frac{n!}{x!(n-x)!} \frac{x!}{y!(x-y)!} p^x (1-p)^{n-x} q^y (1-q)^{x-y}$$

Now, let's pull out all of the constants not related to the index x so we can see what we would like to sum exactly.

$$\frac{n!q^y(1-p)^n(1-q)^{-y}}{y!}\sum_{x=y}^{n}\frac{1}{(n-x)!(x-y)!}p^x(1-p)^{-x}(1-q)^x$$

So the sum we have at hand is not so bad. There is a common motif in performing these sums with factorials: if you have something of the form $\frac{1}{j!k!}$, it may be helpful to rewrite it as $\frac{1}{(j+k)!}\binom{j+k}{j}$. Let's give it a shot:

$$\frac{n!q^y(1-p)^n(1-q)^{-y}}{y!}\cdot\frac{1}{(n-y)!}\sum_{x=y}^{n}\binom{n-y}{x-y}p^x(1-p)^{-x}(1-q)^x$$

And now we see something interesting. We have

$$\sum_{x=y}^{n}\binom{n-y}{x-y}p^x(1-p)^{-x}(1-q)^x=\sum_{x=y}^{n}\binom{n-y}{x-y}\left(\frac{p(1-q)}{1-p}\right)^x$$

Now comes the Binomial Theorem, which states that

$$(1+t)^k=\sum_{j=0}^{k}\binom{k}{j}t^j$$

This motiviates us to try to match the exponent on $\frac{p(1-q)}{1-p}$ with the $x-y$, so we pull out a factor of $\left(\frac{p(1-q)}{1-p}\right)^y$ to get

$$\begin{aligned}\sum_{x=y}^{n}\binom{n-y}{x-y}\left(\frac{p(1-q)}{1-p}\right)^x &= \left(\frac{p(1-q)}{1-p}\right)^y\sum_{x=y}^{n}\binom{n-y}{x-y}\left(\frac{p(1-q)}{1-p}\right)^{x-y}\\ &= \left(\frac{p(1-q)}{1-p}\right)^y\left(1+\frac{p(1-q)}{1-p}\right)^{n-y} && \text{(Used Binomial Theorem here)}\\ &= \frac{p^y(1-q)^y}{(1-p)^y}\cdot\frac{(1-pq)^{n-y}}{(1-p)^{n-y}} && \text{(Fraction simplification)}\end{aligned}$$

The whole goal of this was to clear out the sum. Now that we have cleared out the sum, we can multiply back in the constants we pulled out at the beginning and just do a cancellation exercise

$$\frac{n!q^y\cancel{(1-p)^n}\cancel{(1-q)^{-y}}}{y!(n-y)!}\left(\frac{p^y\cancel{(1-q)^y}}{\cancel{(1-p)^y}}\cdot\frac{(1-pq)^{n-y}}{\cancel{(1-p)^{n-y}}}\right)=\binom{n}{y}(pq)^y(1-pq)^{n-y}$$

Thus, we arrive at the conclusion that $Y\sim\text{Binomial}(n,pq)$.