1 Continuous Computations

Let X be a continuous random variable whose pdf is cx^3 (for some constant c) in the range $0 \le x \le 1$, and is 0 outside this range.

(a) Find c.

(b) Find $\mathbb{P}[1/3 \le X \le 2/3 \mid X \le 1/2]$.

(c) Find $\mathbb{E}(X)$.

(d) Find $\operatorname{var}(X)$.

Solution:

(a) Since our total probability must be equal to 1,

$$\int_0^1 cx^3\,\mathrm{d}x = 1 = \frac{1}{4}cx^4\Big|_{x=0}^1 = \frac{c}{4},$$

so $c = 4$.

(b)

$$\mathbb{P}\left[\frac{1}{3} \le X \le \frac{2}{3} \,\middle|\, X \le \frac{1}{2}\right] = \frac{\mathbb{P}[1/3 \le X \le 2/3 \cap X \le 1/2]}{\mathbb{P}[X \le 1/2]} = \frac{\mathbb{P}[1/3 \le X \le 1/2]}{\mathbb{P}[X \le 1/2]}$$
$$= \frac{\int_{1/3}^{1/2} 4x^3\,\mathrm{d}x}{\int_0^{1/2} 4x^3\,\mathrm{d}x} = \frac{[x^4]_{x=1/3}^{1/2}}{[x^4]_{x=0}^{1/2}} = \frac{(1/2)^4 - (1/3)^4}{(1/2)^4} = \frac{65}{81}.$$

(c)

$$\mathbb{E}(X) = \int_0^1 x \cdot 4x^3\,\mathrm{d}x = \int_0^1 4x^4\,\mathrm{d}x = \left[\frac{4}{5}x^5\right]_{x=0}^1 = \frac{4}{5}.$$

(d)

$$\operatorname{var}(X) = \int_0^1 x^2 \cdot 4x^3\,\mathrm{d}x - \mathbb{E}(X)^2 = \int_0^1 4x^5\,\mathrm{d}x - \left(\frac{4}{5}\right)^2 = \left[\frac{2}{3}x^6\right]_{x=0}^1 - \frac{16}{25} = \frac{2}{75}.$$

2 Lunch Meeting

Alice and Bob agree to try to meet for lunch between 12 PM and 1 PM at their favorite sushi restaurant. Being extremely busy, they are unable to specify their arrival times exactly, and can say only that each of them will arrive (independently) at a time that is uniformly distributed within the hour. In order to avoid wasting precious time, if the other person is not there when they arrive they agree to wait exactly fifteen minutes before leaving. What is the probability that they will actually meet for lunch?

Solution:

Let the random variable A be the time that Alice arrives and the random variable B be the time when Bob arrives. Since A and B are both uniformily distributed, it is helpful to vizualize the distributiion graphically. Consider Figure 1, plotting the space of all outcomes (a,b): The arrival

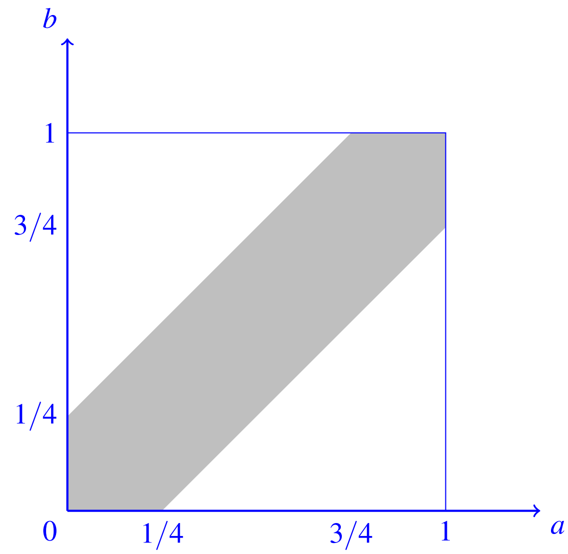

Figure 1: Visualization of joint probability density.

times are uniformily distributed over the box. The shaded region is the set of values (a,b) for which Alice and Bob will actually meet for lunch. Since all points in this square are equally likely, the probably they meet is the ratio of the shaded area to the area of the square. If the area of the square is 1, then the area of the shaded region is

$$1 - 2 \times \left[\frac{1}{2} \times \left(\frac{3}{4}\right)^2\right] = \frac{7}{16},$$

since the area of the white triangle on the upper-left is $(1/2) \cdot (3/4)^2$, and the white triangle on the lower-right has the same area. Therefore, the probability that Alice and Bob actually meet is $7/16$.

3 Exponential Distributions: Lightbulbs

A brand new lightbulb has just been installed in our classroom, and you know the life span of a lightbulb is exponentially distributed with a mean of 50 days.

(a) Suppose an electrician is scheduled to check on the lightbulb in 30 days and replace it if it is broken. What is the probability that the electrician will find the bulb broken?

(b) Suppose the electrician finds the bulb broken and replaces it with a new one. What is the probablity that the new bulb will last at least 30 days?

(c) Suppose the electrician finds the bulb in working condition and leaves. What is the probability that the bulb will last at least another 30 days?

Solution:

(a) Let $X \sim \text{Exponential}(1/50)$ be the time until the bulb is broken. For an exponential random variable with parameter λ, the density function is $f_X(x) = \lambda\,\mathrm{e}^{-\lambda x}$ for $x > 0$. So in this case $\lambda = 1/50$. Thus we can integrate the density to find the probability that the lightbulb broke in the first 30 days:

$$\mathbb{P}[X < 30] = \int_0^{30} \left(\frac{1}{50} \cdot \mathrm{e}^{-x/50}\right) \mathrm{d}x = 1 - \mathrm{e}^{-30/50} = 1 - \mathrm{e}^{-3/5} \approx 0.451.$$

(b) The new bulb's waiting time Y is i.i.d. with the old bulb's. So the answer is

$$\mathbb{P}[Y > 30] = 1 - \mathbb{P}[Y < 30] = 1 - (1 - \mathrm{e}^{-3/5}) = \mathrm{e}^{-3/5} \approx 0.549.$$

(c) The bulb is memoryless, so the probability it will last 60 days given that it has lasted 30 days, is just the probability it will last 30 days:

$$\mathbb{P}[X > 60 \mid X > 30] = \mathbb{P}[X - 30 > 30 \mid X > 30] = \mathbb{P}[X > 30] = \mathrm{e}^{-3/5} \approx 0.549.$$

4 First Exponential to Die

Let X and Y be Exponential(λ_1) and Exponential(λ_2) respectively, independent. What is

$$\mathbb{P}\big(\min(X,Y) = X\big),$$

the probability that the first of the two to die is X?

Solution:

Recall that the CDF of an exponential is $\mathbb{P}[X \le x] = 1 - \exp(-\lambda x)$ for $x \ge 0$.

$$\begin{aligned}\mathbb{P}\big(\min(X,Y) = X\big) &= \mathbb{P}(Y > X) = \int_0^\infty \mathbb{P}(Y > X \mid X = x) f_X(x)\,\mathrm{d}x = \int_0^\infty \mathrm{e}^{-\lambda_2 x} \cdot \lambda_1 \mathrm{e}^{-\lambda_1 x}\,\mathrm{d}x \\ &= -\frac{\lambda_1}{\lambda_1 + \lambda_2} \mathrm{e}^{-(\lambda_1+\lambda_2)x} \Big|_{x=0}^{\infty} = \frac{\lambda_1}{\lambda_1 + \lambda_2}.\end{aligned}$$