CS 70 Discrete Mathematics and Probability Theory
Spring 2019 Babak Ayazifar and Satish Rao **DIS 11B**

1 Continuous Joint Densities

The joint probability density function of two random variables X and Y is given by $f(x,y) = Cxy$ for $0 \leq x \leq 1, 0 \leq y \leq 2$, and 0 otherwise (for a constant C).

(a) Find the constant C that ensures that $f(x,y)$ is indeed a probability density function.

(b) Find $f_X(x)$, the marginal distribution of X

(c) Find the conditional distribution of Y given $X = x$.

(d) Are X and Y independent?

Solution:

(a) Since $f(x,y)$ is a probability density function, it must integrate to 1. Then:

$$1 = \int_0^1 \int_0^2 Cxy\,dy\,dx = \int_0^1 2Cx dx = C$$

Therefore, $C = 1$.

(b) To get the marginal distribution of X, we integrate the joint distribution with respect to Y. So:

$$f_X(x) = \int_0^2 f(x,y)dy = \int_0^2 xydy = 2x$$

This is the marginal distribution for $0 \leq x \leq 1$

(c) The conditional distribution of Y given by

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{xy}{2x} = \frac{y}{2}$$

(d) The conditional distribution of Y given $X = x$ does not depend on x, so they are independent.

Alternatively, you could find the marginal distribution of Y and see it is the same as the conditional distribution of Y:

$$f_Y(y) = \int_0^1 f(x,y)dx = \int_0^1 xydx = \frac{y}{2}$$

Notice that since X and Y are independent, $f_X(x)f_Y(y) = xy = f_{X,Y}(x,y)$, i.e. the product of the marginal distributions is the same as the joint distribution.

2 Arrows

You and your friend are competing in an archery competition. You are a more skilled archer than he is, and the distances of your arrows to the center of the bullseye are i.i.d. Uniform$[0,1]$ whereas his are i.i.d. Uniform$[0,2]$. To even out the playing field, you both agree that you will shoot one arrow and he will shoot two. The arrow closest to the center of the bullseye wins the competition. What is the probability that you will win? *Note*: The distances *from the center of the bullseye* are uniform.

Solution:

Let X be the distance of your arrow to the center and Y that of the closest of your friend's arrows. Then, for $x \in [0,1]$ and $y \in [0,2]$,

$$\mathbb{P}[X > x] = 1 - x \quad \text{and} \quad \mathbb{P}[Y > y] = \left(1 - \frac{y}{2}\right)^2.$$

Hence,

$$f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_X(x) = \frac{\mathrm{d}}{\mathrm{d}x}(1 - \mathbb{P}[X > x]) = -\frac{\mathrm{d}}{\mathrm{d}x}(1 - x) = 1, \qquad x \in [0,1].$$

Also,

$$\mathbb{P}[Y > X \mid X = x] = \left(1 - \frac{x}{2}\right)^2.$$

Thus,

$$\begin{aligned}\mathbb{P}[Y > X] &= \int_0^1 \mathbb{P}(Y > X \mid X = x) f_X(x)\,\mathrm{d}x = \int_0^1 \left(1 - \frac{x}{2}\right)^2 f_X(x)\,\mathrm{d}x = \mathbb{E}\left[\left(1 - \frac{X}{2}\right)^2\right] \\ &= \mathbb{E}\left[1 - X + \frac{X^2}{4}\right] = 1 - \frac{1}{2} + \frac{1}{12} = \frac{7}{12},\end{aligned}$$

since $\mathbb{E}[X] = 1/2$ and $\mathbb{E}[X^2] = \int_0^1 x^2\,\mathrm{d}x = 1/3$.

3 Uniform Distribution

You have two fidget spinners, each having a circumference of 10. You mark one point on each spinner as a needle and place each of them at the center of a circle with values in the range $[0,10)$ marked on the circumference. If you spin both (independently) and let X be the position of the first spinner's mark and Y be the position of the second spinner's mark, what is the probability that $X \geq 5$, given that $Y \geq X$?

Solution:

First we write down what we want and expand out the conditioning:

$$\mathbb{P}[X \geq 5 \mid Y \geq X] = \frac{\mathbb{P}[Y \geq X \cap X \geq 5]}{\mathbb{P}[Y \geq X]}.$$

$\mathbb{P}[Y \geq X] = 1/2$ by symmetry. To find $\mathbb{P}[Y \geq X \cap X \geq 5]$, it helps a lot to just look at the picture of the probability space and use the continuous uniform law $\mathbb{P}[A] = (\text{area of } A)/(\text{area of } \Omega)$. We are interested in the relative area of the region bounded by $x < y < 10$, $5 < x < 10$ to the entire square bounded by $0 < x < 10$, $0 < y < 10$.

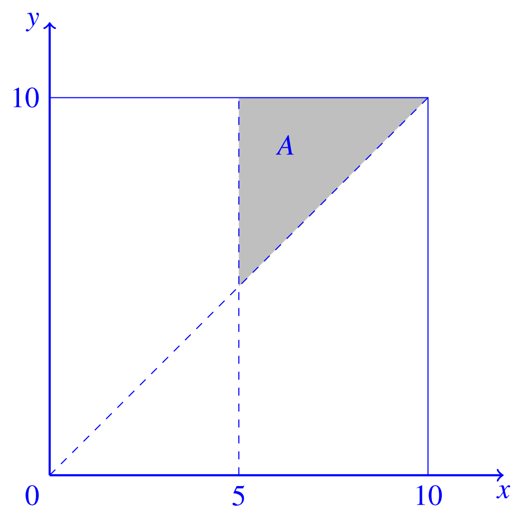

Figure 1: Joint probability density for the spinner.

$$\mathbb{P}[Y \geq X \cap X \geq 5] = \frac{5 \cdot 5/2}{10 \cdot 10} = \frac{1}{8}.$$

So $\mathbb{P}[X \geq 5 \mid Y \geq X] = (1/8)/(1/2) = 1/4$.

4 Darts

Edward and Khalil are playing darts.

Edward's throws are uniformily distributed over the entire dartboard, which has a radius of 10 inches. Khalil has good aim; the distance of his throws from the center of the dartboard follows an exponential distribution with parameter $1/2$.

Say that Edward and Khalil both throw one dart at the dartboard. Let X be the distance of Edward's dart from the center, and Y be the distance of Khalil's dart from the center of the dartboard. What is $\mathbb{P}(X < Y)$, the probability that Edward's throw is closer to the center of the board than Khalil's? Leave your answer in terms of an unevaluated integral.

[*Hint:* X is not uniform over $[0, 10]$. Solve for the distribution of X by first computing the CDF of X, $\mathbb{P}(X < x)$.]

Solution: We are given that $Y \sim \text{Exponential}(1/2)$. We now find the distribution of X by solving for the CDF of X, $\mathbb{P}(X < x)$. To get this, we'll consider the ratio of the area where the distance to the center is less than x, compared to the entire available area. This gives us the following expression:

$$\begin{aligned}\mathbb{P}(X < x) &= \frac{\pi x^2}{\pi 10^2} \\ &= \frac{x^2}{100}\end{aligned}$$

Differentiating gives us the PDF of X, which is given by $f_X(x) = \frac{x}{50}$. Now, we solve for $\mathbb{P}(X < Y)$:

$$\begin{aligned}\mathbb{P}(X < Y) &= \int_{t=0}^{10} f_X(t)\mathbb{P}(Y > t)dt \\ &= \int_{t=0}^{10} \frac{t}{50} e^{-0.5t} dt\end{aligned}$$

Evaulating this integral gives us $\mathbb{P}(X < Y) \approx 0.0767$.