1 Normal Distribution

Recall the following facts about the normal distribution: if $X \sim \mathcal{N}(\mu, \sigma^2)$, then the random variable $Z = (X-\mu)/\sigma$ is standard normal, i.e. $Z \sim \mathcal{N}(0,1)$. There is no closed-form expression for the CDF of the standard normal distribution, so we define $\Phi(z) = \mathbb{P}[Z \le z]$. You may express your answers in terms of $\Phi(z)$.

The average jump of a certain frog is 3 inches. However, because of the wind, the frog does not always go exactly 3 inches. A zoologist tells you that the distance the frog travels is normally distributed with mean 3 and variance $1/4$.

(a) What is the probability that the frog jumps more than 4 inches?

(b) What is the probability that the distance the frog jumps is between 2 and 4 inches?

Solution:

(a) First, we write down the probability we want to find, then transform the probability in order to work with the standard normal.

$$\mathbb{P}[X > 4] = \mathbb{P}[X - 3 > 1] = \mathbb{P}\Big[\frac{X-3}{1/2} > 2\Big] = \mathbb{P}[Z > 2] = 1 - \Phi(2) \approx 0.0228$$

(b) Since the mean of the jump is 3, and the normal distribution is symmetric, we can rewrite the desired probability as

$$\mathbb{P}[2 < X < 4] = 1 - (\mathbb{P}[X > 4] + \mathbb{P}[X < 2]) = 1 - 2 \cdot \mathbb{P}[X > 4].$$

We have computed $\mathbb{P}[X > 4] = 0.0228$ in Part (a), so we can plug this in to get 0.9544.

2 Sum of Independent Gaussians

In this question, we will introduce an important property of the Gaussian distribution: the sum of independent Gaussians is also a Gaussian.

Let X and Y be independent standard Gaussian random variables. Recall that the density of the standard Gaussian is

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{x^2}{2}\Big).$$

(a) What is the joint density of X and Y?

(b) Observe that the joint density of X and Y, $f_{X,Y}(x,y)$, only depends on the quantity x^2+y^2, which is the distance from the origin. In other words, the Gaussian is *rotationally symmetric*. Next, we will try to find the density of $X+Y$. To do this, draw a picture of the Cartesian plane and draw the region $x+y \le c$, where c is a real number of your choice.

(c) Now, rotate your picture clockwise by $\pi/4$ so that the line $X+Y=c$ is now vertical. Redraw your figure. Let X' and Y' denote the random variables which correspond to the $\pi/4$ clockwise rotation of (X,Y) and express the new shaded region in terms of X' and Y'.

(d) By rotational symmetry of the Gaussian, (X',Y') has the same distribution as (X,Y). Argue that $X+Y$ has the same distribution as $\sqrt{2}Z$, where Z is a standard Gaussian. This proves the following important fact: *the sum of independent Gaussians is also a Gaussian*. Notice that $X \sim \mathcal{N}(0,1)$, $Y \sim \mathcal{N}(0,1)$ and $X+Y \sim \mathcal{N}(0,2)$. *In general, if X and Y are independent Gaussians, then $X+Y$ is a Gaussian with mean $\mu_X+\mu_Y$ and variance $\sigma_X^2+\sigma_Y^2$.*

(e) Recall the CLT:

If $\{X_i\}_{i\in\mathbb{N}}$ is a sequence of i.i.d. random variables with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 < \infty$, then:

$$\frac{X_1+\cdots+X_n-n\mu}{\sigma\sqrt{n}} \xrightarrow{\text{in distribution}} \mathcal{N}(0,1) \qquad \text{as } n \to \infty.$$

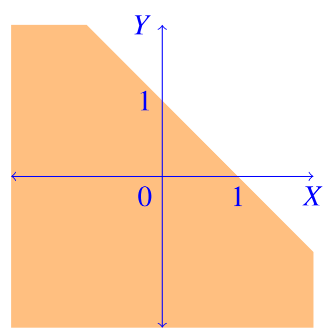
Prove that the CLT holds for the special case when the X_i are i.i.d. $\mathcal{N}(0,1)$.

Solution:

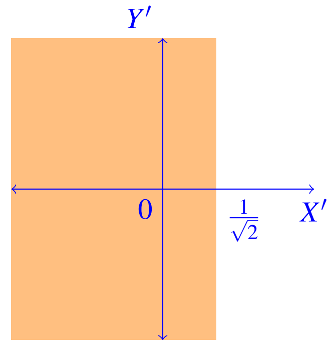
(a) By independence, we have

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = \frac{1}{2\pi}\exp\Big(-\frac{x^2+y^2}{2}\Big).$$

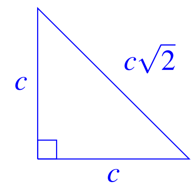
(b) We draw the line for $c=1$.



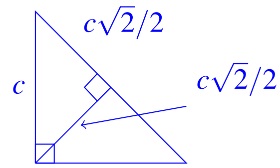
(c) Here is the new figure after the rotation (for $c=1$).



For general $c \in \mathbb{R}$, the new region is $\{X' \le c/\sqrt{2}\}$. To see why, draw the triangle: We want



to find the distance between the origin and the long side of the triangle, and we can do so by adding a diagonal:



(d) We observe that $\mathbb{P}(X+Y \le c) = \mathbb{P}(X' \le c/\sqrt{2}) = \mathbb{P}(\sqrt{2}X' \le c)$, where X' is a standard Gaussian by rotational symmetry, so this proves the claim.

(e) Here, $\mu = 0$ and $\sigma = 1$. So, by the previous part,

$$\frac{X_1 + \cdots + X_n}{\sqrt{n}} \sim \frac{1}{\sqrt{n}} \mathcal{N}(0,n) \sim \mathcal{N}(0,1).$$

3 Hypothesis testing

We would like to test the hypothesis claiming that a coin is fair, i.e. $P(H) = P(T) = 0.5$. To do this, we flip the coin $n = 100$ times. Let Y be the number of heads in $n = 100$ flips of the coin. We decide to reject the hypothesis if we observe that the number of heads is less than $50 - c$ or larger than $50 + c$. However, we would like to avoid rejecting the hypothesis if it is true; we want to keep the probability of doing so less than 0.05. Please determine c. (*Hints: use the central limit theorem to estimate the probability of rejecting the hypothesis given it is actually true.*)

Solution:

NORMAL CUMULATIVE DISTRIBUTION FUNCTION

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Let X_i be the random variable denoting the result of the i-th flip:

$$X_i = \begin{cases} 1 & \text{if the } i\text{-th flip is head,} \\ 0 & \text{if the } i\text{-th flip is tail.} \end{cases}$$

Then we have $Y = \sum_{i=1}^{n} X_i$. If the hypothesis is true, then $\mu = \mathbb{E}[X_i] = \frac{1}{2}$ and $\sigma^2 = \text{var}(X_i) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. By central limit theorem, we know that

$$P\left(\frac{Y - n\mu}{\sqrt{n\sigma^2}} \leq z\right) \approx \Phi(z)$$
$$P\left(\frac{Y - 100 \cdot \frac{1}{2}}{\sqrt{100 \cdot \frac{1}{4}}} \leq z\right) \approx \Phi(z)$$
$$P\left(\frac{Y - 50}{5} \leq z\right) \approx \Phi(z)$$

where

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \mathrm{d}x.$$

We will reject the hypothesis when $|Y - 50| > c$. We also want $P(|Y - 50| > c) < 0.05$, or equivalently $P(|Y - 50| \leq c) > 0.95$. We have

$$P(|Y - 50| \leq c) = P\left(\frac{|Y - 50|}{5} \leq \frac{c}{5}\right) \approx 2\Phi(\frac{c}{5}) - 1.$$

The reason this is $\approx 2\Phi(\frac{c}{5}) - 1$ is because the probability we are looking for is the probability that Y is within $\frac{c}{5}$ standard deviations of the mean. By an area argument, we can see that this is $\Phi(\frac{c}{5}) - (1 - \Phi(\frac{c}{5})) = 2\Phi(\frac{c}{5}) - 1$. Let $2\Phi(\frac{c}{5}) - 1 = 0.95$, so $\Phi(\frac{c}{5}) = 0.975$ or $\frac{c}{5} = 1.96$. That is $c = 9.8$ flips. So we see that if we observe more that $50 + 10 = 60$ or less than $50 - 10 = 40$ heads, we can reject the hypothesis.