1 Allen's Umbrellas

Every morning, Allen walks from his home to Soda, and every evening, Allen walks from Soda to his home. Suppose that Allen has two umbrellas in his possession, but he sometimes leaves his umbrellas behind. Specifically, before leaving from his home or Soda, he checks the weather. If it is raining outside, he will bring his umbrella (that is, if there is an umbrella where he currently is). If it is not raining outside, he will forget to bring his umbrella. Assume that the probability of rain is p.

We will model this as a Markov chain. Let $\mathscr{X} = \{0,1,2\}$ be the set of states, where the state i represents the number of umbrellas in his current location. Determine if the distribution of X_n converges to the invariant distribution, and compute the invariant distribution. Determine the long-term fraction of time that Allen will walk through rain with no umbrella.

Solution:

Suppose Allen is in state 0. Then, Allen has no umbrellas to bring, so with probability 1 Allen arrives at a location with 2 umbrellas. That is,

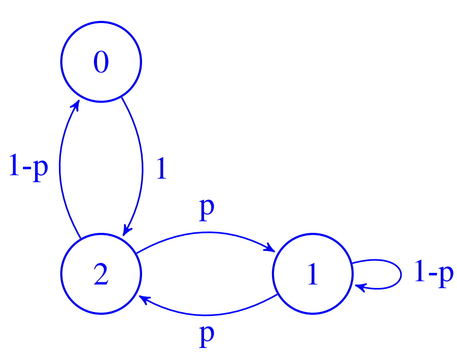
$$\mathbb{P}[X_{n+1} = 2 \mid X_n = 0] = 1.$$

Suppose Allen is in state 1. With probability p, it rains and Allen brings the umbrella, arriving at state 2. With probability $1-p$, Allen forgets the umbrella, so Allen arrives at state 1.

$$\mathbb{P}[X_{n+1} = 2 \mid X_n = 1] = p, \qquad \mathbb{P}[X_{n+1} = 1 \mid X_n = 1] = 1-p$$

Suppose Allen is in state 2. With probability p, it rains and Allen brings the umbrella, arriving at state 1. With probability $1-p$, Allen forgets the umbrella, so Allen arrives at state 0.

$$\mathbb{P}[X_{n+1} = 1 \mid X_n = 2] = p, \qquad \mathbb{P}[X_{n+1} = 0 \mid X_n = 2] = 1-p$$



We summarize this with the transition matrix

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix}.$$

Observe that the transition matrix has non-zero element in its diagonal, which means the minimum number of steps to transit to state 1 from itself is one. Thus this transition matrix is irreducible and aperiodic, so it converges to its invariant distribution. To solve for the distribution, we set $\pi P = \pi$, or $\pi(P - I) = 0$. This yields the balance equations

$$\begin{bmatrix} \pi(0) & \pi(1) & \pi(2) \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 0 & -p & p \\ 1-p & p & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}.$$

As usual, one of the equations is redundant. We replace the last column by the normalization condition $\pi(0) + \pi(1) + \pi(2) = 1$.

$$\begin{bmatrix} \pi(0) & \pi(1) & \pi(2) \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 0 & -p & 1 \\ 1-p & p & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

Now solve for the distribution:

$$\begin{bmatrix} \pi(0) & \pi(1) & \pi(2) \end{bmatrix} = \frac{1}{3-p} \begin{bmatrix} 1-p & 1 & 1 \end{bmatrix}$$

The invariant distribution also tells us the long-term fraction of time that Allen spends in each state. We can see that Allen spends a fraction $(1-p)/(3-p)$ of his time with no umbrella in his location, so the long-term fraction of time in which he walks through rain is $p(1-p)/(3-p)$.

2 Reflecting Random Walk

Alice starts at vertex 0 and wishes to get to vertex n. When she is at vertex 0 she has a probability of 1 of transitioning to vertex 1. For any other vertex i, there is a probability of $1/2$ of transitioning to $i+1$ and a probability of $1/2$ of transitioning to $i-1$.

(a) What is the expected number of steps Alice takes to reach vertex n? Write down the hitting-time equations, but do not solve them yet.

(b) Solve the hitting-time equations. [*Hint*: Let R_i denote the expected number of steps to reach vertex n starting from vertex i. As a suggestion, try writing R_0 in terms of R_1; then, use this to express R_1 in terms of R_2; and then use this to express R_2 in terms of R_3, and so on. See if you can notice a pattern.]

Solution:

Formulate hitting time equations; the hard part is solving them. R_i represents the expected number of steps to get to vertex n starting from vertex i. In particular, $R_n = 0$ and we are interested in calculating R_0. We have the equations:

$$\begin{aligned}
R_0 &= 1 + R_1, \\
R_1 &= 1 + \frac{1}{2}R_0 + \frac{1}{2}R_2, \\
&\vdots \\
R_i &= 1 + \frac{1}{2}R_{i-1} + \frac{1}{2}R_{i+1}, \\
&\vdots \\
R_{n-1} &= 1 + \frac{1}{2}R_{n-2} + \frac{1}{2}R_n.
\end{aligned}$$

Plug in $R_0 = 1 + R_1$ to the second equation: $R_1 = 1 + 1/2 + (1/2)R_1 + (1/2)R_2$ which then implies $R_1 = 3 + R_2$. In fact, if $R_i = k + R_{i+1}$, then

$$R_{i+1} = 1 + \frac{1}{2}R_i + \frac{1}{2}R_{i+2} = 1 + \frac{k}{2} + \frac{1}{2}R_{i+1} + \frac{1}{2}R_{i+2},$$

which, after moving $(1/2)R_{i+1}$ to the left and multiplying by two, implies $R_{i+1} = k + 2 + R_{i+2}$.

Therefore, $R_0 = 1 + R_1 = 1 + 3 + R_2 = 1 + 3 + 5 + R_3 = \cdots = 1 + 3 + \cdots + 2n - 1 + R_n$ and since $R_n = 0$, we have $R_0 = n^2$.

3 Predicament

Three men are on a boat with cigarettes, but they have no lighter. What do they do?

Solution:

One man throws his cigarette off the boat. The boat is now a cigarette lighter!

4 Which Envelope?

You have two envelopes in front of you containing cash. You know that one envelope contains twice as much money as the other envelope (the amount of money in an envelope is an integer). You are allowed to pick one envelope and see how much cash is inside, and then based on this information, you can decide to switch envelopes or stick with the envelope you already have.

Can you come up with a strategy which will allow you to pick the envelope with more money, with probability strictly greater than $1/2$?

Solution:

Surprisingly, the answer is yes.

First, we are going to choose a random positive integer z. To accomplish this, we flip coins! Let n be the number of flips you need before you see heads. Then, set $z = 2^{n-1} + 0.5$ (okay we lied; this is not an integer, but the 0.5 is just there to break ties later on).

Here is the strategy: look at how much cash is in the envelope you pick. If the amount of cash in the envelope exceeds z, then keep the envelope; otherwise, switch to the other envelope.

Suppose z is smaller than the amount of money in either envelope. Then, you will always keep your original envelope, but since the envelope you chose in the first place is equally likely to be the more lucrative envelope, you still have a $1/2$ probability of choosing the right envelope.

A similar analysis holds if z is greater than the amount of money in either envlope; here you will always switch envelopes, and you have a $1/2$ probability of choosing the right envelope.

However, what happens if z happens to land in between the values of the envelopes? If you initially chose the envelope with less money, then you will switch to the better envelope; if you initially chose the envlope with more money, you will stick with the better envelope. So, in this case, you are guaranteed to end up with the better envelope!

Since z has a positive probability of landing between the values of the envelopes, this strategy gives you a probability of choosing the better envelope which is strictly better than $1/2$.