CS 70 Discrete Mathematics and Probability Theory
Fall 2019 Alistair Sinclair and Yun S. Song **DIS 2**

1 Induction

Prove the following using induction:

(a) For all natural numbers $n \geq 1$, if $A = \begin{bmatrix} -2 & -9 \\ 1 & 4 \end{bmatrix}$, then $A^n = \begin{bmatrix} 1-3n & -9n \\ n & 1+3n \end{bmatrix}$.

(b) For real numbers a_i where $-1 < a_i \leq 0$, $i \in \mathbb{N}$, $\prod_{i=0}^{i=n}(1+a_i) \geq 1 + \sum_{i=0}^{i=n} a_i$.

2 Binary Numbers

Prove that every positive integer n can be written in binary. In other words, prove that we can write

$$n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + c_0 \cdot 2^0,$$

where $k \in \mathbb{N}$ and $c_k \in \{0,1\}$.

3 Stable Marriage

The following questions refer to stable marriage instances with n men and n women, answer True/-False or provide an expression as requested.

(a) For $n = 2$, or any 2-man, 2-woman stable marriage instance, man A has the same optimal and pessimal woman. (True or False.)

(b) In any stable marriage instance, in the pairing the Stable Marriage Algorithm produces there is some man who gets his favorite woman (the first woman on his preference list). (True or False.)

(c) In any stable marriage instance with n men and women, if every man has a different favorite woman, a different second favorite, a different third favorite, and so on, and every woman has the same preference list, how many days does it take for Stable Marriage Algorithm to finish? (Form of Answer: An expression that may contain n.)

(d) Consider a stable marriage instance with n men and n women, and where all men have the same preference list, and all women have different favorite men, and different second-favorite men, and so on. How many days does the Stable Marriage Algorithm take to finish? (Form of Answer: An expression that may contain n.)

(e) It is possible for a stable pairing to have a man A and a woman 1 be paired if A is 1's least preferred choice and 1 is A's least preferred choice. (True or False.)

4 Universal Preference

Suppose that preferences in a stable marriage instance are universal: all n men share the preferences $W_1 > W_2 > \cdots > W_n$ and all women share the preferences $M_1 > M_2 > \cdots > M_n$.

(a) What pairing do we get from running the algorithm with men proposing? Can you prove this happens for all n?

(b) What pairing do we get from running the algorithm with women proposing?

(c) What does this tell us about the number of stable pairings?