CS 70 Discrete Mathematics and Probability Theory
Spring 2021 **DIS 2B**

1 True or False

(a) Any pair of vertices in a tree are connected by exactly one path.

(b) A simple graph obtained by adding an edge between two vertices of a tree creates a cycle.

(c) Adding an edge in a connected graph creates exactly one new cycle.

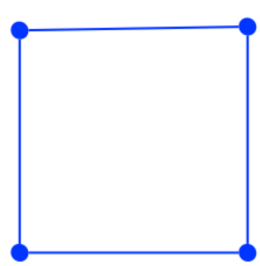
Solution:

(a) **True.**
Pick any pair of vertices x, y. We know there is a path between them since the graph is connected. We will prove that this path is unique by contradiction: Suppose there are two distinct paths from x to y. At some point (say at vertex a) the paths must diverge, and at some point (say at vertex b) they must reconnect. So by following the first path from a to b and the second path in reverse from b to a we get a cycle. This gives the necessary contradiction.

(b) **True.**
Pick any pair of vertices x, y not connected by an edge. We prove that adding the edge $\{x,y\}$ will create a cycle. From part (a), we know that there is a unique path between x and y. Therefore, adding the edge $\{x,y\}$ creates a cycle obtained by following the path from x to y, then following the edge $\{x,y\}$ from y back to x.

Adding an edge between a pair of vertices x, y connected by an edge creates a graph that is not simple. So, we do not need to consider this case. To be sure, the condition is not really needed as the existence of two edges beteen x and y also creates a cycle.

(c) **False.**
In the following graph adding an edge creates two cycles.



2 Coloring Trees

Prove that all trees with at least 2 vertices are *bipartite*: the vertices can be partitioned into two groups so that every edge goes between the two groups.

[*Hint:* Use induction on the number of vertices.]

Solution:

Proof using induction on the number of vertices n.

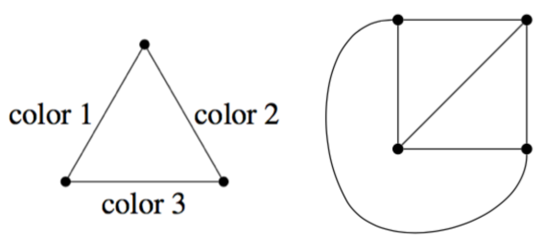
Base case $n = 2$. A tree with two vertices has only one edge and is a bipartite graph by partitioning the two vertices into two separate parts.

Inductive hypothesis. Assume that all trees with k vertices for an arbitrary $k \geq 2$ is bipartite.

Inductive step. Consider a tree $T = (V,E)$ with $k+1$ vertices. We know that every tree must have at least two leaves, so remove one leaf u and the edge connected to u, say edge e. The resulting graph $T - u$ is a tree with k vertices and is bipartite by the inductive hypothesis. Thus there exists a partitioning of the vertices $V = R \cup L$ such that there does not exist an edge that connects two vertices in L or two vertices in R. Now when we add u back to the graph. If edge e connects u with a vertex in L then let $L' = L$ and $R' = R \cup \{u\}$. On the other hand if edge e connects u with a vertex in R then let $L' = L \cup \{u\}$ and $R' = R$. L' and R' gives us the required partition to show that T is bipartite. This completes the inductive step and hence by induction we get that all trees with at least 2 vertices are bipartite.

3 Edge Colorings

An edge coloring of a graph is an assignment of colors to edges in a graph where any two edges incident to the same vertex have different colors. An example is shown on the left.



(a) Show that the 4 vertex complete graph above can be 3 edge colored. (Use the numbers $1,2,3$ for colors. A figure is shown on the right.)

(b) Prove that any graph with maximum degree $d \geq 1$ can be edge colored with $2d-1$ colors.

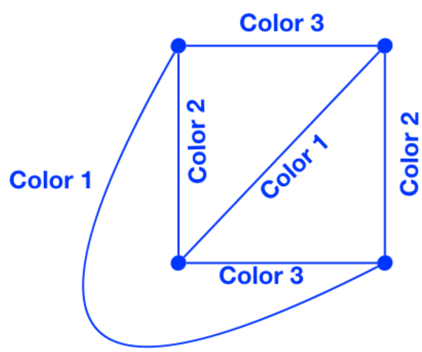
(c) Show that a tree can be edge colored with d colors where d is the maximum degree of any vertex.

Solution:

(a) Three color a triangle u_1, u_2, u_3 where (u_1, u_2) is colored 1, (u_2, u_3) is colored 2, and (u_3, u_1) is colored 3. This is a valid 3 coloring as the edges are all colored differently.

Consider adding a fourth vertex v, the incident edges must be colored differently and each incident edge (v, u_i) needs to be colored differently from the edges incident to u_i. That is, one can color (v, u_1) with 2 as it is not incident to the edge colored 2 and that color is avalaible. Similarly one can color edge (v, u_2) with color 3 and (v, u_3) with color 1.

Another proof is simply provide a coloring which is below.



(b) We will use induction on the number of edges n in the graph to prove the statement: If a graph G has $n \geq 0$ edges and the maximum degree of any vertex is d, then G can be colored with $2d-1$ colors.

Base case (n = 0). If there are no edges in the graph, then there is nothing to be colored and the statement holds trivially.

Inductive hypothesis. Suppose for $n = k \geq 0$, the statement holds.

Inductive step. Consider a graph G with $n = k+1$ edges. Remove an edge of your choice, say e from G. Note that in the resulting graph the maximum degree of any vertes is $d' \leq d$. By the inductive hypothesis, we can color this graph using $2d'-1$ colors and hence with $2d-1$ colors too. The removed edge is incident to two vertices each of which is incident to at most $d-1$ other edges, and thus at most $2(d-1) = 2d-2$ colors are unavailable for edge e. Thus, we can color edge e without any conflicts. This proves the statement for $n = k+1$ and hence by induction we get that the statement holds for all $n \geq 0$.

(c) We will use induction on the number of vertices n in the tree to prove the statement: For a tree with $n \geq 1$ vertices, if the maximum degree of any vertex is d, then the tree can be colored with d colors.

Base case (n=1). If there is only one vertex, then there are no edges to color, and thus can be colored with 0 colors.

Inductive hypothesis. Suppose the statement holds for $n = k \geq 1$.

Inductive Step. Remove any leaf v of your choice from the tree. We can then color the remaining tree with d colors by the inductive hypothesis. For any neighboring vertex u of vertex v, the degree of u is at most $d-1$ since we removed the edge $\{u, v\}$ along with the vertex v. Thus its incident edges use at most $d-1$ colors and there is a color available for coloring the edge {u,v}. This completes the inductive step and by induction we have that the statement holds for all $n \geq 1$.

4 Hypercubes

The vertex set of the n-dimensional hypercube $G = (V, E)$ is given by $V = \{0,1\}^n$ (recall that $\{0,1\}^n$ denotes the set of all n-bit strings). There is an edge between two vertices x and y if and only if x and y differ in exactly one bit position. These problems will help you understand hypercubes.

(a) Draw 1-, 2-, and 3-dimensional hypercubes and label the vertices using the corresponding bit strings.

(b) Show that for any $n \geq 1$, the n-dimensional hypercube is bipartite.

Solution:

(a) The three hypercubes are a line, a square, and a cube, respectively. See also note 6 for pictures.

(b) Consider the vertices with an even number of 0 bits and the vertices with an odd number of 0 bits. Each vertex with an even number of 0 bits is adjacent only to vertices with an odd number of 0 bits, since each edge represents a single bit change (either a 0 bit is added by flipping a 1 bit, or a 0 bit is removed by flipping a 0 bit). Let L be the set of the vertices with an even number of 0 bits and let R be the vertices with an odd number of 0 bits, then no two adjacent vertices will belong to the same set.

Alternate solution (using induction and coloring):

It may be simpler to that a graph being 2-colorable is the same as being bipartite. Now, the argument is easier to state. First the base case is a hypercube with two vertices which is clearly two-colorable. Then notice, switching the colors in a two-coloring is still valid as if endpoints are differently colored, switching leaves them differently colored. Now, recursively one two colors the two subcubes the same, and then switches the colors in one subcube. The internal to subcube edges are fine by induction. The edges across are fine as the corresponding vertices are differently colored due to the switching.