1 Hilbert's Hotel

You don't have any summer plans, so you decide to spend a few months working for a magical hotel with a countably infinite number of rooms. The rooms are numbered according to the natural numbers, and all the rooms are currently occupied. Assume that guests don't mind being moved from their current room to a new one, so long as they can get to the new room in a finite amount of time (i.e. guests can't be moved into a room infinitely far from their current one).

(a) A new guest arrives at the hotel. All the current rooms are full, but your manager has told you never to turn away a guest. How could you accommodate the new guest by shuffling other guests around? What if you instead had k guest arrive, for some fixed, positive $k \in \mathbb{Z}$?

(b) Unfortunately, just after you've figured out how to accommodate your first $k+1$ guests, a countably infinite number of guests arrives in town on an infinitely long train. The guests on the train are sitting in seats numbered according to the natural numbers. How could you accommodate all the new guests?

(c) Thanks to a (literally) endless stream of positive TripAdvisor reviews, word of the infinite hotel gets around quickly. Soon enough you find out that a countably infinite number of trains have arrived in town. Each is of infinite length, and carries a countably infinite number of passengers. How would you accommodate all the new passengers?

Solution:

(a) Shift all guests into the room number that is k greater than their current room number. So for a guest in room i move him/her to room $i+k$. Then place the k new guests in the k first rooms in the hotel which will now be unoccupied.

(b) Place all existing guests in room $2i$ where i is their current room number. Place all the new guests in room $2j+1$ where j is their seat number on the train.

(c) **Solution 1**: We first set up a bijection between the newly arriving guests and the set $\mathbb{N} \times \mathbb{N}$. Notice that each guest has an "address": his/her train number i and his/her seat number j. Let this guest be mapped to (i, j). It is clear that this is a bijection.

We know from Lecture Note 10 that the set $\mathbb{N} \times \mathbb{N}$ is countable (via the spiral method) and hence there is a bijection from $\mathbb{N}^2$ to $\mathbb{N}$. Thus the newly arriving guests can be enumerated and considered as if arriving in a single infinite length train with their corresponding seat numbers given by the enumeration. This reduces to the same exact problem as the previous part! Therefore, we can accommodate these guests.

Solution 2: Place all existing guests in room 2^i where i is their current room number. Assign the $(k+2)$th prime, p_{k+2}, to the kth train (e.g. the 0th train will be assigned the 2nd prime, 3). We then place each new guest in room p_{k+2}^{j+1}, where j is the seat number of the new guest on that train.

This works because any power of a prime p will not have any prime factors other than p.

Yes, there will be plenty of empty rooms, but that's okay because every guest will still have somewhere to stay.

2 Countability Practice

(a) Do $(0,1)$ and $\mathbb{R}_+ = (0,\infty)$ have the same cardinality? If so, either give an explicit bijection (and prove that it is a bijection) or provide an injection from $(0,1)$ to $(0,\infty)$ and an injection from $(0,\infty)$ to $(0,1)$ (so that by Cantor-Bernstein theorem the two sets will have the same cardinality). If not, then prove that they have different cardinalities.

(b) Is the set of strings over the English alphabet countable? (Note that the strings may be arbitrarily long, but each string has finite length. Also the strings need not be real English words.) If so, then provide a method for enumerating the strings. If not, then use a diagonalization argument to show that the set is uncountable.

(c) Consider the previous part, except now the strings are drawn from a countably infinite alphabet $\mathscr{A}$. Does your answer from before change? Make sure to justify your answer.

Solution:

(a) Yes, they have the same cardinality.
Explicit bijection: Consider the bijection $f : (0,1) \to (0,\infty)$ given by

$$f(x) = \frac{1}{x} - 1.$$

We show that f is a bijection by proving separately that it is one-to-one and onto. The function f is one-to-one: suppose that $f(x) = f(y)$. Then,

$$\begin{aligned} \frac{1}{x} - 1 &= \frac{1}{y} - 1, \\ \frac{1}{x} &= \frac{1}{y}, \\ x &= y. \end{aligned}$$

Hence, f is one-to-one.

The function f is onto: take any $y \in (0,\infty)$. Let $x = 1/(1+y)$. Note that $x \in (0,1)$. Then,

$$f(x) = \frac{1}{1/(1+y)} - 1 = 1 + y - 1 = y,$$

so f maps x to y. Hence, f is onto.

We have exhibited a bijection from $(0,1)$ to $(0,\infty)$, so they have the same cardinality. (In fact, they are both uncountable.)

Indirect bijection: The injection from $(0,1)$ to $(0,\infty)$ is trivial; consider the function $f : (0,1) \to (0,\infty)$ given by

$$f(x) = x.$$

It is easy to see that f is injective.

For the other way, consider the function $g : (0,\infty) \to (0,1)$ given by

$$g(x) = \frac{1}{x+1}.$$

To see that g is injective, suppose $g(x) = g(y)$. Then

$$\frac{1}{x+1} = \frac{1}{y+1} \implies x = y.$$

Hence g is injective. Thus we have an injective function from $(0,1)$ to $(0,\infty)$ and an injective function from $(0,\infty)$ to $(0,1)$. By Cantor-Bernstein theorem there exists a bijection from $(0,1)$ to $(0,\infty)$ and hence they have the same cardianlity.

(b) Countable. The English language has a finite alphabet (52 characters if you count only lower-case and upper-case letters, or more if you count special symbols – either way, the alphabet is finite).

We will now enumerate the strings in such a way that each string appears exactly once in the list. We will use the same trick as used in Lecture note 10 to enumerate the elements of $\{0,1\}^*$ We get our bijection by setting $f(n)$ to be the n-th string in the list. List all strings of length 1 in lexicographic order, and then all strings of length 2 in lexicographic order, and then strings of length 3 in lexicographic order, and so forth. Since at each step, there are only finitely many strings of a particular length ℓ, any string of finite length appears in the list. It is also clear that each string appears exactly once in this list.

(c) No, the strings are still countable. Let $\mathscr{A} = \{a_1, a_2, \dots\}$ denote the alphabet. (We are making use of the fact that the alphabet is countably infinite when we assume there is such an enumeration.) We will provide two solutions:

Alternative 1: We will enumerate all the strings similar to that in part (b), although the enumeration requires a little more finesse. Notice that if we tried to list all strings of length 1, we would be stuck forever, since the alphabet is infinite! On the other hand, if we try to restrict our alphabet and only print out strings containing the first character $a \in \mathscr{A}$, we would also have a similar problem: the list

$$a, aa, aaa, \dots$$

also does not end.

The idea is to restrict *both* the length of the string and the characters we are allowed to use:

(a) List all strings containing only a_1 which are of length at most 1.

(b) List all strings containing only characters in $\{a_1, a_2\}$ which are of length at most 2 and have not been listed before.

(c) List all strings containing only characters in $\{a_1, a_2, a_3\}$ which are of length at most 3 and have not been listed before.

(d) Proceed onwards.

At each step, we have restricted ourselves to a finite alphabet with a finite length, so each step is guaranteed to terminate. To show that the enumeration is complete, consider any string s of length ℓ; since the length is finite, it can contain at most ℓ distinct a_i from the alphabet. Let k denote the largest index of any a_i which appears in s. Then, s will be listed in step $\max(k, \ell)$, so it appears in the enumeration. Further, since we are listing only those strings that have not appeared before, each string appears exactly once in the listing.

Alternative 2: We will encode the strings into ternary strings. Recall that we used a similar trick in Lecture note 10 to show that the set of all polynomials with natural coefficients is countable. Suppose, for example, we have a string: $S = a_5a_2a_7a_4a_6$. Corresponding to each of the characters in this string, we can write its index as a binary string: $(101, 10, 111, 100, 110)$. Now, we can construct a ternary string where "2" is inserted as a separator between each binary string. Thus we map the string S to a ternary string: 101210211121002110. It is clear that this mapping is injective, since the original string S can be uniquely recovered from this ternary string. Thus we have an injective map to $\{0, 1, 2\}^*$. From Lecture note 10, we know that the set $\{0, 1, 2\}^*$ is countable, and hence the set of all strings with finite length over $\mathscr{A}$ is countable.

3 Counting Functions

Are the following sets countable or uncountable? Prove your claims.

(a) The set of all functions f from $\mathbb{N}$ to $\mathbb{N}$ such that f is non-decreasing. That is, $f(x) \leq f(y)$ whenever $x \leq y$.

(b) The set of all functions f from $\mathbb{N}$ to $\mathbb{N}$ such that f is non-increasing. That is, $f(x) \geq f(y)$ whenever $x \leq y$.

Solution:

(a) Uncountable: Let us assume the contrary and proceed with a diagonalization argument. If there are countably many such function we can enumerate them as

	0	1	2	3	...
f_0	$f_0(0)$	$f_0(1)$	$f_0(2)$	$f_0(3)$	...
f_1	$f_1(0)$	$f_1(1)$	$f_1(2)$	$f_1(3)$	...
f_2	$f_2(0)$	$f_2(1)$	$f_2(2)$	$f_2(3)$	...
f_3	$f_3(0)$	$f_3(1)$	$f_3(2)$	$f_3(3)$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Now go along the diagonal and define f such that $f(x) > f_x(x)$ and $f(y) > f(x)$ if $y > x$, which is possible because at step k we only need to find a number $\in \mathbb{N}$ greater than all the $f_j(j)$ for $j \in \{0,\ldots,k\}$. This function differs from each f_i and therefore cannot be on the list, hence the list does not exhaust all non-decreasing functions. As a result, there must be uncountably many such functions.

Alternative Solution: Look at the subset $\mathscr{S}$ of strictly increasing functions. Any such f is uniquely identified by its image which is an infinite subset of $\mathbb{N}$. But the set of infinite subsets of $\mathbb{N}$ is uncountable. This is because the set of all subsets of $\mathbb{N}$ is uncountable, and the set of all finite subsets of $\mathbb{N}$ is countable. So $\mathscr{S}$ is uncountable and hence the set of all non-decreasing functions must be too.

Alternative Solution 2: We can inject the set of infinitely long binary strings into the set of non-decreasing functions as follows. For any infinitely long binary string b, let $f(n)$ be equal to the number of 1's appearing in the first n-digits of b. It is clear that the function f so defined is non-decreasing. Also, since the function f is uniquely defined by the infinitely long binary string, the mapping from binary strings to non-decreasing functions is injective. Since the set of infinite binary strings is uncountable, and we produced an injection from that set to the set of non-decreasing functions, that set must be uncountable as well.

(b) Countable: Let D_n be the subset of non-increasing functions for which $f(0) = n$. Any such function must stop decreasing at some point (because $\mathbb{N}$ has a smallest number), so there can only be finitely many (at most n) points $X_f = \{x_1,\ldots,x_k\}$ at which f decreases. Let y_i be the amount by which f decreases at x_i, then f is fully described by $\{(x_1,y_1),\ldots,(x_k,y_k),$ $(-1,0),\ldots,(-1,0)\} \in \mathbb{N}^n = \mathbb{N}\times\mathbb{N}\times\cdots\times\mathbb{N}$ (n times), where we padded the k values associated with f with $n-k$ $(-1,0)$s. In Lecture note 10, we have seen that $\mathbb{N}\times\mathbb{N}$ is countable by the spiral method. Using it repeatedly, we get $\mathbb{N}^{(2^l)}$ is countable for all $l \in \mathbb{N}$. This gives us that $\mathbb{N}^n$ is countably for any finite n (because $\mathbb{N}^n \subset \mathbb{N}^{(2^l)}$ where l is such that $2^l \geq n$). Hence D_n is countable. Since each set D_n is countable we can enumerate it. Map an element of D_n to (n, j) where j is the label of that element produced by the enumeration of D_n. This produces an injective map from $\cup_{n\in\mathbb{N}} D_n$ to $\mathbb{N}\times\mathbb{N}$ and we know that $\mathbb{N}\times\mathbb{N}$ is countable from Lecture note 10 (via spiral method). Now the set of all non-increasing functions is $\cup_{i\in\mathbb{N}} D_n$, and thus countable.