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1 Miscellaneous Logic

(a) Let the statement, $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})\ G(x,y)$, be true for predicate $G(x,y)$.

For each of the following statements, decide if the statement is certainly true, certainly false, or possibly true, and justify your solution. (If possibly true, provide a specific example where the statement is false and a specific example where the statement is true.)

(i) $G(3,4)$

(ii) $(\forall x \in \mathbb{R})\ G(x,3)$

(iii) $\exists y\ G(3,y)$

(iv) $\forall y\ \neg G(3,y)$

(v) $\exists x\ G(x,4)$

(b) Give an expression using terms involving $\vee, \wedge$ and $\neg$ which is true if and only if exactly one of X, Y, and Z is true.

Solution:

(a) (i) **Possibly true.**
The statement only guarantees there exists some y such that $G(3,y)$ is true, not that $G(3,4)$ in particular is true, so this is possibly but not necessarily true.
Choose $G(x,y)$ to be always true and statement is true.
Choose $G(x,y)$ to be $x > y$ to be false.

(ii) **Possibly true.**
In the same vein as the previous part, we are guaranteed the existence of *a* y for each x, not that $G(x,3)$ is necessarily true.
Choose $G(x,y)$ to be always true and statement is true.
Choose $G(x,y)$ to be $x > y$ to be false.

(iii) **True.**
The original statement is that for every x, there is a y where $G(x,y)$ is true which implies that for $x = 3$, there is a y where $G(x,y)$ is true.

(iv) **False.**
This is the negation of the statement above.

(v) **Possibly true.**
This is similar to part b. We don't have information about $G(x,4)$ specifically - only that there exists a y for x such that $G(x,y)$ is true.
Choose $G(x,y)$ to be always true and statement is true.
Choose $G(x,y)$ to be $y \neq 4$.

(b) $(X \wedge \neg Y \wedge \neg Z) \vee (\neg X \wedge Y \wedge \neg Z) \vee (\neg X \wedge \neg Y \wedge Z)$
There are 3 cases in which exactly one of X, Y, and Z are true, and the cases are joined with "or"s because we only require one case to occur. Within each case, we fix one of X, Y, and Z to be true and the other two to be false. Since the three cases are mutually exclusive, at least one of them being true is equivalent to exactly one of them being true.

2 Propositional Practice

Convert the following English sentences into propositional logic and the following propositions into English. State whether or not each statement is true with brief justification.

(a) There is a real number which is not rational.

(b) All integers are natural numbers or are negative, but not both.

(c) If a natural number is divisible by 6, it is divisible by 2 or it is divisible by 3.

(d) $(\forall x \in \mathbb{R})\ (x \in \mathbb{C})$

(e) $(\forall x \in \mathbb{Z})\ (((2 \mid x) \vee (3 \mid x)) \implies (6 \mid x))$

(f) $(\forall x \in \mathbb{N})\ ((x > 7) \implies ((\exists a, b \in \mathbb{N})\ (a + b = x)))$

Solution:

(a) $(\exists x \in \mathbb{R})\ (x \notin \mathbb{Q})$, or equivalently $(\exists x \in \mathbb{R})\ \neg(x \in \mathbb{Q})$. This is true, and we can use π as an example to prove it.

(b) $(\forall x \in \mathbb{Z})\ (((x \in \mathbb{N}) \vee (x < 0)) \wedge \neg((x \in \mathbb{N}) \wedge (x < 0)))$. This is true, since we define the naturals to contain all integers which are not negative.

(c) $(\forall x \in \mathbb{N})\ ((6 \mid x) \implies ((2 \mid x) \vee (3 \mid x)))$. This is true, since any number divisible by 6 can be written as $6k = (2 \cdot 3)k = 2(3k)$, meaning it must also be divisible by 2.

(d) All real numbers are complex numbers. This is true, since any real number x can equivalently be written as $x + 0i$.

(e) Any integer that is divisible by 2 or 3 is also divisible by 6. This is false–2 provides the easiest counterexample. Note that this statement is false even though its converse (part c) is true.

(f) If a natural number is larger than 7, it can be written as the sum of two other natural numbers.
This is trivially true, since we can take $a = x$ and $b = 0$.

(Aside: this is a refererence to the very weak Goldback Conjecture (`https://xkcd.com/1310/`).)

3 Prove or Disprove

(a) $(\forall n \in \mathbb{N})$ if n is odd then $n^2 + 2n$ is odd.

(b) $(\forall x, y \in \mathbb{R})$ $\min(x, y) = (x + y - |x - y|)/2$.

(c) $(\forall a, b \in \mathbb{R})$ if $a + b \leq 10$ then $a \leq 7$ or $b \leq 3$.

(d) $(\forall r \in \mathbb{R})$ if r is irrational then $r + 1$ is irrational.

(e) $(\forall n \in \mathbb{Z}^+)$ $10n^2 > n!$.

Solution:

(a) **Answer**: True.
Proof: We will use a direct proof. Assume n is odd. By the definition of odd numbers, $n = 2k + 1$ for some natural number k. Substituting into the expression $n^2 + 2n$, we get $(2k + 1)^2 + 2 \times (2k + 1)$. Simplifying the expression yields $4k^2 + 8k + 3$. This can be rewritten as $2 \times (2k^2 + 4k + 1) + 1$. Since $2k^2 + 4k + 1$ is a natural number, by the definition of odd numbers, $n^2 + 2n$ is odd.
Alternatively, we could also factor the expression to get $n(n + 2)$. Since n is odd, $n + 2$ is also odd. The product of 2 odd numbers is also an odd number. Hence $n^2 + 2n$ is odd.

(b) **Answer**: True.
Proof: We will use a proof by cases. We know the following about the absolute value function for real number z.

$$|z| = \begin{cases} z, & z \geq 0 \\ -z, & z < 0 \end{cases}$$

Case 1: $x < y$. This means $|x - y| = y - x$. Substituting this into the formula on the right hand side, we get

$$\frac{x + y - y + x}{2} = x = \min(x, y).$$

Case 2: $x \geq y$. This means $|x - y| = x - y$. Substituting this into the formula on the right hand side, we get

$$\frac{x + y - x + y}{2} = y = \min(x, y).$$

(c) **Answer**: True.
Proof: We will use a proof by contraposition. Suppose that $a > 7$ and $b > 3$ (note that this is equivalent to $\neg\big(a \leq 7 \vee b \leq 3\big)$). Since $a > 7$ and $b > 3$, $a + b > 10$ (note that $a + b > 10$ is equivalent to $\neg\big(a + b \leq 10\big)$). Thus, if $a + b \leq 10$, then $a \leq 7$ or $b \leq 3$.

(d) **Answer**: True.
Proof: We will use a proof by contraposition. Assume that $r+1$ is rational. Since $r+1$ is rational, it can be written in the form a/b where a and b are integers with $b \neq 0$. Then r can be written as $(a-b)/b$. By the definition of rational numbers, r is a rational number, since both $a-b$ and b are integers, with $b \neq 0$. By contraposition, if r is irrational, then $r+1$ is irrational.

(e) **Answer**: False.
Proof: We will use proof by counterexample. Let $n = 6$. $10 \times 6^2 = 360$. $6! = 720$. Since $10n^2 < n!$, the claim is false.

4 Preserving Set Operations

For a function f, define the image of a set X to be the set $f(X) = \{y \mid y = f(x) \text{ for some } x \in X\}$. Define the inverse image or preimage of a set Y to be the set $f^{-1}(Y) = \{x \mid f(x) \in Y\}$. Prove the following statements, in which A and B are sets. By doing so, you will show that inverse images preserve set operations, but images typically do not.

Hint: For sets X and Y, $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$. To prove that $X \subseteq Y$, it is sufficient to show that $(\forall x)\ ((x \in X) \implies (x \in Y))$.

(a) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(b) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

(c) $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$.

(d) $f(A \cup B) = f(A) \cup f(B)$.

(e) $f(A \cap B) \subseteq f(A) \cap f(B)$, and give an example where equality does not hold.

(f) $f(A \setminus B) \supseteq f(A) \setminus f(B)$, and give an example where equality does not hold.

Solution:

In order to prove equality $A = B$, we need to prove that A is a subset of B, $A \subseteq B$ and that B is a subset of A, $B \subseteq A$. To prove that LHS is a subset of RHS we need to prove that if an element is a member of LHS then it is also an element of the RHS.

(a) Suppose x is such that $f(x) \in A \cup B$. Then either $f(x) \in A$, in which case $x \in f^{-1}(A)$, or $f(x) \in B$, in which case $x \in f^{-1}(B)$, so in either case we have $x \in f^{-1}(A) \cup f^{-1}(B)$. This proves that $f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$.
Now, suppose that $x \in f^{-1}(A) \cup f^{-1}(B)$. Suppose, without loss of generality, that $x \in f^{-1}(A)$. Then $f(x) \in A$, so $f(x) \in A \cup B$, so $x \in f^{-1}(A \cup B)$. The argument for $x \in f^{-1}(B)$ is the same. Hence, $f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$.

(b) Suppose x is such that $f(x) \in A \cap B$. Then $f(x)$ lies in both A and B, so x lies in both $f^{-1}(A)$ and $f^{-1}(B)$, so $x \in f^{-1}(A) \cap f^{-1}(B)$. So $f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$.

Now, suppose that $x \in f^{-1}(A) \cap f^{-1}(B)$. Then, x is in both $f^{-1}(A)$ and $f^{-1}(B)$, so $f(x) \in A$ and $f(x) \in B$, so $f(x) \in A \cap B$, so $x \in f^{-1}(A \cap B)$. So $f^{-1}(A) \cap f^{-1}(B) \subseteq f^{-1}(A \cap B)$.

(c) Suppose x is such that $f(x) \in A \setminus B$. Then, $f(x) \in A$ and $f(x) \notin B$, which means that $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$, which means that $x \in f^{-1}(A) \setminus f^{-1}(B)$. So $f^{-1}(A \setminus B) \subseteq f^{-1}(A) \setminus f^{-1}(B)$.

Now, suppose that $x \in f^{-1}(A) \setminus f^{-1}(B)$. Then, $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$, so $f(x) \in A$ and $f(x) \notin B$, so $f(x) \in A \setminus B$, so $x \in f^{-1}(A \setminus B)$. So $f^{-1}(A) \setminus f^{-1}(B) \subseteq f^{-1}(A \setminus B)$.

(d) Suppose that $x \in A \cup B$. Then either $x \in A$, in which case $f(x) \in f(A)$, or $x \in B$, in which case $f(x) \in f(B)$. In either case, $f(x) \in f(A) \cup f(B)$, so $f(A \cup B) \subseteq f(A) \cup f(B)$.

Now, suppose that $y \in f(A) \cup f(B)$. Then either $y \in f(A)$ or $y \in f(B)$. In the first case, there is an element $x \in A$ with $f(x) = y$; in the second case, there is an element $x \in B$ with $f(x) = y$. In either case, there is an element $x \in A \cup B$ with $f(x) = y$, which means that $y \in f(A \cup B)$. So $f(A) \cup f(B) \subseteq f(A \cup B)$.

(e) Suppose $x \in A \cap B$. Then, x lies in both A and B, so $f(x)$ lies in both $f(A)$ and $f(B)$, so $f(x) \in f(A) \cap f(B)$. Hence, $f(A \cap B) \subseteq f(A) \cap f(B)$.

Consider when there are elements $a \in A$ and $b \in B$ with $f(a) = f(b)$, but A and B are disjoint. Here, $f(a) = f(b) \in f(A) \cap f(B)$, but $f(A \cap B)$ is empty (since $A \cap B$ is empty).

(f) Suppose $y \in f(A) \setminus f(B)$. Since y is not in $f(B)$, there are no elements in B which map to y. Let x be any element of A that maps to y; by the previous sentence, x cannot lie in B. Hence, $x \in A \setminus B$, so $y \in f(A \setminus B)$. Hence, $f(A) \setminus f(B) \subseteq f(A \setminus B)$.

Consider when $B = \{0\}$ and $A = \{0, 1\}$, with $f(0) = f(1) = 0$. One has $A \setminus B = \{1\}$, so $f(A \setminus B) = \{0\}$. However, $f(A) = f(B) = \{0\}$, so $f(A) \setminus f(B) = \varnothing$.

5 Hit or Miss?

State which of the proofs below is correct or incorrect. For the incorrect ones, please explain clearly where the logical error in the proof lies. Simply saying that the claim or the induction hypothesis is false is *not* a valid explanation of what is wrong with the proof. You do not need to elaborate if you think the proof is correct.

(a) **Claim:** For all positive numbers $n \in \mathbb{R}$, $n^2 \geq n$.

Proof. The proof will be by induction on n.
Base Case: $1^2 \geq 1$. It is true for $n = 1$.
Inductive Hypothesis: Assume that $n^2 \geq n$.
Inductive Step: We must prove that $(n+1)^2 \geq n+1$. Starting from the left hand side,

$$\begin{aligned}(n+1)^2 &= n^2 + 2n + 1 \\ &\geq n + 1.\end{aligned}$$

Therefore, the statement is true. $\square$

(b) **Claim:** For all negative integers n, $(-1)+(-3)+\cdots+(2n+1) = -n^2$.

Proof. The proof will be by induction on n.
Base Case: $-1 = -(-1)^2$. It is true for $n = -1$.
Inductive Hypothesis: Assume that $(-1)+(-3)+\cdots+(2n+1) = -n^2$.
Inductive Step: We need to prove that the statement is also true for $n-1$ if it is true for n, that is, $(-1)+(-3)+\cdots+(2(n-1)+1) = -(n-1)^2$. Starting from the left hand side,

$$\begin{aligned}(-1)+(-3)+\cdots+(2(n-1)+1) &= ((-1)+(-3)+\cdots+(2n+1))+(2(n-1)+1)\\ &= -n^2+(2(n-1)+1) \quad \text{(Inductive Hypothesis)}\\ &= -n^2+2n-1\\ &= -(n^2-2n+1)\\ &= -(n-1)^2.\end{aligned}$$

Therefore, the statement is true. $\square$

(c) **Claim:** For all nonnegative integers n, $2n = 0$.

Proof. We will prove by strong induction on n.
Base Case: $2 \times 0 = 0$. It is true for $n = 0$.
Inductive Hypothesis: Assume that $2k = 0$ for all $0 \le k \le n$.
Inductive Step: We must show that $2(n+1) = 0$. Write $n+1 = a+b$ where $0 < a, b \le n$. From the inductive hypothesis, we know $2a = 0$ and $2b = 0$, therefore,

$$2(n+1) = 2(a+b) = 2a+2b = 0+0 = 0.$$

The statement is true. $\square$

Solution:

(a) Note that n is a real number. The proof is incorrect because it does not consider $0 < n < 1$, for which the claim is false. Also, by the way it is set up, it can only cover integers for $n \ge 1$.

(b) The proof is correct. The base case starts from the correct, identifiable end point, then the inductive step successfully proves that the statement continues to be true towards $-\infty$.

(c) The proof is incorrect. When $n = 0$, we cannot write $n+1 = 1 = a+b$ where $0 < a, b \le n = 0$.

6 Badminton Ranking

A team of n ($n \geq 2$) badminton players held a tournament, where every person plays with every other person exactly once, and there are no ties. Prove by induction that after the tournament, we can arrange the n players in a sequence, so that every player in the sequence has won against the person immediately to the right of him.

Solution:

Denote the n players by P_1 through P_n. Define a sequence of arrangement that satisfies the given condition to be a *valid* sequence.

For $n = 2$, either P_1 wins against P_2 or P_2 wins against P_1, and a valid sequence is given by P_1P_2 in the first case, and P_2P_1 in the second case.

Assume that we can construct a valid sequence for $n = k$. For $n = k+1$, we can arrange the first k players in a valid sequence. Denote this sequence by $P_{i_1} \cdots P_{i_k}$, where $i_1 \cdots i_k$ is a permutation of 1 through k. Now we just need to insert P_{k+1} in the appropriate position. We proceed as follows:

1. If P_{k+1} wins against P_{i_1}, we can put P_{k+1} in the beginning of the sequence; otherwise we go to the next step.
2. If P_{k+1} wins against P_{i_2}, we can put P_{k+1} between P_{i_1} and P_{i_2}; otherwise we go to the next step.
3. ...

k. If P_{k+1} wins against P_{i_k}, we can put P_{k+1} between $P_{i_{k-1}}$ and P_{i_k}; otherwise we simply put P_{k+1} at the end of the sequence.

In all these situations, we can successfully construct a valid sequence of P_1 through P_{k+1}. Hence the induction step is proven, and the original claim follows.