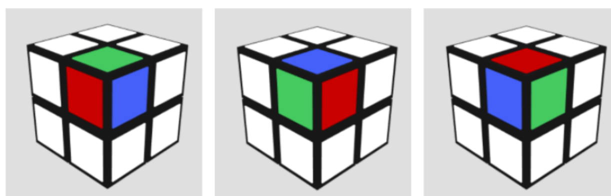
CS 70 Spring 2019 — Discrete Mathematics and Probability Theory Course Notes — Homework 7

1 Rubik's Cube Scrambles

We wish to count the number of ways to scramble a $2 \times 2 \times 2$ Rubik's Cube, and take a quick look at the $3 \times 3 \times 3$ cube. Leave your answer as an expression (rather than trying to evaluate it to get a specific number).

(a) The $2 \times 2 \times 2$ Rubik's Cube is composed of 8 "corner pieces" arranged in a $2 \times 2 \times 2$ cube. How many ways can we assign all the corner pieces a position?

(b) Each corner piece has three distinct colors on it, and so can also be oriented three different ways once it is assigned a position (see figure below). How many ways can we *assemble* (assign each piece a position and orientation) a $2 \times 2 \times 2$ Rubik's Cube?



Three orientations of a corner piece

(c) The previous part assumed we can take apart pieces and assemble them as we wish. But certain configurations are unreachable if we restrict ourselves to just turning the sides of the cube. What this means for us is that if the orientations of 7 out of 8 of the corner pieces are determined, there is only 1 valid orientation for the eighth piece. Given this, how many ways are there to *scramble* (as opposed to *assemble*) a $2 \times 2 \times 2$ Rubik's Cube?

(d) We decide to treat scrambles that differ only in overall positioning (e.g. flipped upside-down or rotated but otherwise unaltered) as the same scramble. Then we overcounted in the previous part! How does this new condition change your answer to the previous part?

(e) Now consider the $3 \times 3 \times 3$ Rubik's Cube. In addition to 8 corner pieces, we now have 12 "edge" pieces, each of which can take 2 orientations. How many ways can we *assemble* a $3 \times 3 \times 3$ Rubik's Cube?

Solution:

(a) 8! ways. This is just the number of permutations of 8 objects.

(b) $8! \cdot 3^8$. Each of eight pieces has 3 possible orientations, to add onto the 8! permutations from part (a).

(c) $8! \cdot 3^7$. We divide the previous part by three. We can still choose the orientation of 7 of 8 corner pieces, but the eighth is then fixed.

(d) For each scramble, there are $6 \cdot 4 = 24$ ways to position/rotate it. We can see this by first choosing one of 6 faces to be on "top", then choosing one of four rotations with that particular face on top. So now we have $\frac{1}{24} \cdot 8! \cdot 3^7$.

(e) $8! \cdot 3^8 \cdot 12! \cdot 2^{12}$. We use the same procedure on the edge pieces as we do the corner pieces.

2 Counting, Counting, and More Counting

The only way to learn counting is to practice, practice, practice, so here is your chance to do so. For this problem, you do not need to show work that justifies your answers. We encourage you to leave your answer as an expression (rather than trying to evaluate it to get a specific number).

(a) How many ways are there to arrange n 1s and k 0s into a sequence?

(b) A bridge hand is obtained by selecting 13 cards from a standard 52-card deck. The order of the cards in a bridge hand is irrelevant.
How many different 13-card bridge hands are there? How many different 13-card bridge hands are there that contain no aces? How many different 13-card bridge hands are there that contain all four aces? How many different 13-card bridge hands are there that contain exactly 6 spades?

(c) Two identical decks of 52 cards are mixed together, yielding a stack of 104 cards. How many different ways are there to order this stack of 104 cards?

(d) How many 99-bit strings are there that contain more ones than zeros?

(e) An anagram of FLORIDA is any re-ordering of the letters of FLORIDA, i.e., any string made up of the letters F, L, O, R, I, D, and A, in any order. The anagram does not have to be an English word.
How many different anagrams of FLORIDA are there? How many different anagrams of ALASKA are there? How many different anagrams of ALABAMA are there? How many different anagrams of MONTANA are there?

(f) How many different anagrams of ABCDEF are there if: (1) C is the left neighbor of E; (2) C is on the left of E (and not necessarily E's neighbor)

(g) We have 9 balls, numbered 1 through 9, and 27 bins. How many different ways are there to distribute these 9 balls among the 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).

(h) We throw 9 identical balls into 7 bins. How many different ways are there to distribute these 9 balls among the 7 bins such that no bin is empty? Assume the bins are distinguishable (e.g., numbered 1 through 7).

(i) How many different ways are there to throw 9 identical balls into 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).

(j) There are exactly 20 students currently enrolled in a class. How many different ways are there to pair up the 20 students, so that each student is paired with one other student?

(k) How many solutions does $x_0 + x_1 + \cdots + x_k = n$ have, if each x must be a non-negative integer?

(l) How many solutions does $x_0 + x_1 = n$ have, if each x must be a *strictly positive* integer?

(m) How many solutions does $x_0 + x_1 + \cdots + x_k = n$ have, if each x must be a *strictly positive* integer?

Solution:

(a) $\binom{n+k}{k}$

(b) We have to choose 13 cards out of 52 cards, so this is just $\binom{52}{13}$.

We now have to choose 13 cards out of 48 non-ace cards. So this is $\binom{48}{13}$.

We now require the four aces to be present. So we have to choose the remaining 9 cards in our hand from the 48 non-ace cards, and this is $\binom{48}{9}$.

We need our hand to contain 6 out of the 13 spade cards, and 7 out of the 39 non-spade cards, and these choices can be made separately. Hence, there are $\binom{13}{6}\binom{39}{7}$ ways to make up the hand.

(c) If we consider the 104! rearrangements of 2 identical decks, since each card appears twice, we would have overcounted each distinct rearrangement. Consider any distinct rearrangement of the 2 identical decks of 52 cards and see how many times this appears among the rearrangement of 104 cards where each card is treated as different. For each identical pair (such as the two Ace of spades), there are two ways they could be permuted among each other (since $2! = 2$). This holds for each of the 52 pairs of identical cards. So the number 104! overcounts the actual number of rearrangements of 2 identical decks by a factor of 2^{52}. Hence, the actual number of rearrangements of 2 identical decks is $104!/2^{52}$.

(d) **Answer 1:** There are $\binom{99}{k}$ 99-bit strings with k ones and $99-k$ zeros. We need $k > 99-k$, i.e. $k \geq 50$. So the total number of such strings is $\sum_{k=50}^{99}\binom{99}{k}$.
This expression can however be simplified. Since $\binom{99}{k} = \binom{99}{99-k}$, we have

$$\sum_{k=50}^{99}\binom{99}{k} = \sum_{k=50}^{99}\binom{99}{99-k} = \sum_{l=0}^{49}\binom{99}{l}$$

by substituting $l = 99-k$. Now $\sum_{k=50}^{99}\binom{99}{k} + \sum_{l=0}^{49}\binom{99}{l} = \sum_{m=0}^{99}\binom{99}{m} = 2^{99}$. Hence, $\sum_{k=50}^{99}\binom{99}{k} = (1/2)\cdot 2^{99} = 2^{98}$.

Answer 2: Since the answer from above looked so simple, there must have been a more elegant way to arrive at it. Since 99 is odd, no 99-bit string can have the same number of zeros and ones. Let A be the set of 99-bit strings with more ones than zeros, and B be the set of 99-bit strings with more zeros than ones. Now take any 99-bit string x with more ones than zeros i.e. $x \in A$. If all the bits of x are flipped, then you get a string y with more zeros than ones, and so $y \in B$. This operation of bit flips creates a one-to-one and onto function (called a bijection) between A and B. Hence, it must be that $|A| = |B|$. Every 99-bit string is either in A or in B, and since there are 2^{99} 99-bit strings, we get $|A| = |B| = (1/2) \cdot 2^{99}$. The answer we sought was $|A| = 2^{98}$.

(e) This is the number of ways of rearranging 7 distinct letters and is $7!$.

In this 6 letter word, the letter A is repeated 3 times while the other letters appear once. Hence, the number $6!$ overcounts the number of different anagrams by a factor of $3!$ (which is the number of ways of permuting the 3 A's among themselves). Hence, there are $6!/3!$ different anagrams.

In this 7 letter word, the letter A is repeated 4 times while the other letters appear once. Hence, the number $7!$ overcounts the number of different anagrams by a factor of $4!$ (which is the number of ways of permuting the 4 A's among themselves). Hence, there are $7!/4!$ anagrams.

In this 7 letter word, the letter A and N are each repeated 2 times while the other letters appear once. Hence, the number $7!$ overcounts the number of different anagrams by a factor of $2! \times 2!$ (one factor of $2!$ for the number of ways of permuting the 2 A's among themselves and another factor of $2!$ for the number of ways of permuting the 2 N's among themselves). Hence, there are $7!/(2!)^2$ different anagrams.

(f) (1) We consider CE is a new letter X, then the question becomes counting the rearranging of 5 distinct letters, and is $5!$. (2) Let A be the set of all the rearranging of ABCDEF with C on the left side of E, and B be the set of all the rearranging of ABCDEF with C on the right side of E. $|A \bigcup B| = 6!, |A \cap B| = 0$. There is a bijection between A and B by construct a operation of exchange the position of C and E. Thus $|A| = |B| = 6!/2$.

(g) Each ball has a choice of which bin it should go to. So each ball has 27 choices and the 9 balls can make their choices separately. Hence, there are 27^9 ways.

(h) **Answer 1:** Since each bin is required to be non-empty, let's throw one ball into each bin at the outset. Now we have 2 identical balls left which we want to throw into 7 distinguishable bins. There are 2 cases to consider:
Case 1: The 2 balls land in the same bin. This gives 7 ways.
Case 2: The 2 balls land in different bins. This gives $\binom{7}{2}$ ways of choosing 2 out of the 7 bins for the balls to land in. Note that it is *not* 7×6 since the balls are identical and so there is no order on them.
Summing up the number of ways from both cases, we get $7 + \binom{7}{2}$ ways.

Answer 2: Since each bin is required to be non-empty, let's throw one ball into each bin at the outset. Now we have 2 identical balls left which we want to throw into 7 distinguishable bins. From class (see notes 10), we already saw that the number of ways to put k identical balls into n distinguishable bins is $\binom{n+k-1}{k}$. Taking $k = 2$ and $n = 7$, we get $\binom{8}{2}$ ways to do this.
EASY EXERCISE: Can you give an expression for the number of ways to put k identical balls into n distinguishable bins such that no bin is empty?

(i) Since there is no restriction on how many balls a bin needs to have, this is just the problem of throwing k identical balls into n distinguishable bins, which can be done in $\binom{n+k-1}{k}$ ways. Here $k = 9$ and $n = 27$, so there are $\binom{35}{9}$ ways.

(j) **Answer 1:** Let's number the students from 1 to 20. Student 1 has 19 choices for her partner. Let i be the smallest index among students who have not yet been assigned partners. Then no matter what the value of i is (in particular, i could be 2 or 3), student i has 17 choices for her partner. The next smallest indexed student who doesn't have a partner now has 15 choices for her partner. Continuing in this way, the number of pairings is $19 \times 17 \times 15 \times \cdots \times 1 = \prod_{i=1}^{10}(2i-1)$.

Answer 2: Arrange the students numbered 1 to 20 in a line. There are 20! such arrangements. We pair up the students at positions $2i-1$ and $2i$ for i ranging from 1 to 10. You should be able to see that the 20! permutations of the students doesn't miss any possible pairing. However, it counts every different pairing multiple times. Fix any particular pairing of students. In this pairing, the first pair had freedom of 10 positions in any permutation that generated it, the second pair had a freedom of 9 positions in any permutation that generated it, and so on. There is also the freedom for the elements within each pair i.e. in any student pair (x,y), student x could have appeared in position $2i-1$ and student y could have appeared in position $2i$ and also vice versa. This gives 2 ways for each of the 10 pairs. Thus, in total, these freedoms cause $10! \times 2^{10}$ of the 20! permutations to give rise to this particular pairing. This holds for each of the different pairings. Hence, 20! overcounts the number of different pairings by a factor of $10! \times 2^{10}$. Hence, there are $20!/(10! \cdot 2^{10})$ pairings.

Answer 3: In the first step, pick a pair of students from the 20 students. There are $\binom{20}{2}$ ways to do this. In the second step, pick a pair of students from the remaining 18 students. There are $\binom{18}{2}$ ways to do this. Keep picking pairs like this, until in the tenth step, you pick a pair of students from the remaining 2 students. There are $\binom{2}{2}$ ways to do this. Multiplying all these, we get $\binom{20}{2}\binom{18}{2}\ldots\binom{2}{2}$. However, in any particular pairing of 20 students, this pairing could have been generated in 10! ways using the above procedure depending on which pairs in the pairing got picked in the first step, second step, ..., tenth step. Hence, we have to divide the above number by 10! to get the number of different pairings. Thus there are $\binom{20}{2}\binom{18}{2}\ldots\binom{2}{2}/10!$ different pairings of 20 students.

You may want to check for yourself that all three methods are producing the same integer, even though they are expressed very differently.

(k) $\binom{n+k}{k}$. There is a bijection between a sequence of n ones and k plusses and a solution to the equation: x_0 is the number of ones before the first plus, x_1 is the number of ones between the first and second plus, etc. A key idea is that if a bijection exists between two sets they must be the same size, so counting the elements of one tells us how many the other has.

(l) $n-1$. It's easy just to enumerate the solutions here. x_0 can take values $1, 2, \ldots, n-1$ and this uniquely fixes the value of x_1. So, we have $n-1$ ways to do this. But, this is just an example of the more general question below.

(m) $\binom{(n-(k+1))+k}{k} = \binom{n-1}{k}$. By subtracting 1 from all $k+1$ variables, and $k+1$ from the total required, we reduce it to problem with the same form as the previous problem. Once we have a solution to that we reverse the process, and adding 1 to all the non-negative variables gives us positive variables.

3 Divisor Graph Colorings

Define G where we have $V = \{2,3,4,5,6,7,8,9\}$, and we add an edge between vertex i and vertex j if i divides j, or j divides i.

(a) Explain why we cannot vertex-color G with only 2 colors.

(b) How many ways can we vertex-color G with 3 colors?

Solution:

(a) The vertices 2, 4, and 8 form a length-3 cycle, which cannot be colored.

(b) 432. Vertices 5 and 7 can each take one of three colors. So can vertex 2. Then vertex 4 must take on of two colors, and vertex 8 (being connected to both 2 and 4) is forced to take on a particular color. Vertex 6 (being connected to vertex 2) then has choice of two colors. Vertex 3 (being connected to vertex 6) then has choice of two colors. Vertex 9 (being connected to vertex 3) then has choice of two colors as well. This is a total of $3 \cdot 3 \cdot 3 \cdot 2 \cdot 1 \cdot 2 \cdot 2 \cdot 2 = 432$ colorings.

4 Vacation Time

After a number of complaints, the Dunder Mifflin Paper Company has decided on the following rule for vacation leave for the next year (365 days): Every employee must take exactly one vacation leave of 4 consecutive days, one vacation leave of 5 consecutive days and one vacation leave of 6 consecutive days within the year, with the property that any two of the vacation leaves have a gap of at least 7 days between them. In how many ways can an employee arrange their vacation time? (The vacation policy resets every year, so there is no need to worry about leaving a gap between this year and next year's vacations).

Solution:

We can consider each of the consecutive leaves of n_i vacation days (with i from 1 to 3) as one vacation block of n_i days. Let us first decide when in the year we want to have our 3 vacation blocks, and then consider the ordering of the vacations.

We can append 7 days of work after the first two vacation blocks to create two blocks of n_1+7 and n_2+7 days, so that no matter where we place these blocks in the year, the second vacation will always come at least 7 days after the first, and the third vacation will always come at least 7 days after the second.

Then we have $365-4-5-6-7-7=336$ days remaining, and 3 blocks to place within those days. So we have $336+3$ total blocks and days, giving us $\binom{339}{3}$ ways to choose three places in the year to have vacation blocks.

Now we need to consider which order we'll take the vacations. Since we have 3 choices for the first vacation, 2 for the second and 1 for the third, we have 3! orderings for vacations.

This leaves us a total of $\binom{339}{3}\cdot 3!$ total ways to arrange our vacations in the year.

5 Story Problems

Prove the following identities by combinatorial argument:

(a) $\binom{2n}{2}=2\binom{n}{2}+n^2$

(b) $n^2=2\binom{n}{2}+n$

(c) $\sum_{k=0}^{n} k\binom{n}{k}=n2^{n-1}$
Hint: Consider how many ways there are to pick groups of people ("teams") and then a representative ("team leaders").

(d) $\sum_{k=j}^{n}\binom{n}{k}\binom{k}{j}=2^{n-j}\binom{n}{j}$
Hint: Consider a generalization of the previous part.

Solution:

(a) The left hand side is the number of ways to choose two elements out of $2n$. Counting in another way, we first divide the $2n$ elements (arbitrarily) into two sets of n elements. Then we consider three cases: either we choose both elements out of the first n-element set, both out of the second n-element set, or one element out of each set. The number of ways we can do each of these things is $\binom{n}{2}$, $\binom{n}{2}$, and n^2, respectively. Since these three cases are mutually exclusive and cover all the possibilities, summing them must give the same number as the left hand side. This completes the proof.

(b) LHS: There are n movies. Choose the best-rated movie and the best-selling movie. There are n choices for each title.

RHS: Choose 2 distinct movies, permute them in 2 ways. But what if the best-rated movie is the best selling movie? Then we're just choosing one movie, and there are n ways of doing that.

(c) RHS: From n people, pick one team-leader and some (possibly empty) subset of other people on his team.

LHS: First pick k people on the team, then pick the leader among them.

(d) RHS: Form a team as follows: Pick j leaders from n people. Then pick some (possibly empty) subset of the remaining people.

LHS: First pick $k \geq j$ people on the team, then pick the j leaders among them.

6 Fermat's Wristband

Let p be a prime number and let k be a positive integer. We have beads of k different colors, where any two beads of the same color are indistinguishable.

(a) We place p beads onto a string. How many different ways are there construct such a sequence of p beads with up to k different colors?

(b) How many sequences of p beads on the string are there that use at least two colors?

(c) Now we tie the two ends of the string together, forming a wristband. Two wristbands are equivalent if the sequence of colors on one can be obtained by rotating the beads on the other. (For instance, if we have $k = 3$ colors, red (R), green (G), and blue (B), then the length $p = 5$ necklaces RGGBG, GGBGR, GBGRG, BGRGG, and GRGGB are all equivalent, because these are all rotated versions of each other.)

How many non-equivalent wristbands are there now? Again, the p beads must not all have the same color. (Your answer should be a simple function of k and p.)

[*Hint*: Think about the fact that rotating all the beads on the wristband to another position produces an identical wristband.]

(d) Use your answer to part (c) to prove Fermat's little theorem.

Solution:

(a) k^p. For each of the p beads, there are k possibilities for its colors. Therefore, by the first counting principle, there are k^p different sequences.

(b) $k^p - k$. You can have k sequences of a beads with only one color.

(c) Since p is prime, rotating any sequence by less than p spots will produce a new sequence. As in, there is no number x smaller than p such that rotating the beads by x would cause the pattern to look the same. So, every pattern which has more than one color of beads can be rotated to form $p-1$ other patterns. So the total number of patterns equivalent with some bead sequence is p. Thus, the total number of non-equivalent patterns are $(k^p - k)/p$.

(d) $(k^p - k)/p$ must be an integer, because from the previous part, it is the number of ways to count something. Hence, $k^p - k$ has to be divisible by p, i.e., $k^p \equiv k \pmod{p}$, which is Fermat's Little Theorem.