CS 70 Spring 2019 — Discrete Mathematics and Probability Theory — Satish Rao and Babak Ayazifar — HW 8

1 Five Coins

We toss a coin five times.

(a) For the first three parts, order matters in the outcome. How many different outcomes are possible?

(b) How many different outcomes are possible with exactly 3 heads?

(c) How many different outcomes are possible with 3 or more heads? Justify your answer with a symmetry argument.

(d) For the next three parts, assume that the coin is unbiased. What is the probability of getting the outcome TTHHH? What is the probability of getting the outcome THHHH?

(e) What's the probability of getting at least one heads?

(f) What's the probability of getting 3 or more heads?

(g) For the next three parts, assume that the coin is biased with probability of heads being $\frac{2}{3}$. What is the probability of getting the outcome TTHHH? What is the probability of getting the outcome THHHH?

(h) What's the probability of getting at least one heads?

(i) What's the probability of getting 3 or more heads?

Solution:

We toss a coin five times.

(a) Since for each coin toss, we can have either heads or tails, we have 2^5 total possible outcomes.

(b) Since we know that we have exactly 3 heads, what distinguishes the outcomes is at which point these heads occurred. There are 5 possible places for the heads to occur, and we need to choose 3 of them, giving us the following result: $\binom{5}{3}$.

(c) We can use the same approach from part (b), but since we are asking for 3 or more, we need to consider the cases of exactly 4 heads, and exactly 5 heads as well. This gives us the result as: $\binom{5}{3}+\binom{5}{4}+\binom{5}{5}=16$.
To see why the number is exactly half of the total number of outcomes, denote the set of

outcomes that has 3 or more heads as A. If we flip over every coin in each outcome in set A, we get all the outcomes that has 2 or less head. Denote the new set as A'. Then we know that A and A' have the same size and they together cover the whole sample space. Therefore, $|A| = |A'|$ and $|A| + |A'| = 2^5$, which gives $|A| = 2^5/2$.

(d) Since each coin toss is an independent event, the probability of each of the coin tosses is $\frac{1}{2}$ making the probability of this outcome $\frac{1}{2^5}$. This holds for both cases since both heads and tails have the same probability.

(e) We will use the complementary event, which is the event of getting no heads. The probability of getting no heads is the probability of getting all tails. This event has a probability of $\frac{1}{2^5}$ by a similar argument to the previous part. Since we are asking for the probability of getting at least one heads, our final result is: $1 - \frac{1}{2^5}$.

(f) Since each outcome in this probability space is equally likely, we can divide the number of outcomes where there are 3 or more heads by the total number of outcomes to give us: $\frac{\binom{5}{3}+\binom{5}{4}+\binom{5}{5}}{2^5}$

(g) By using the same idea of independence we get for TTHHH: $\frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{2^3}{3^5}$

For THHHH, we get:

$\frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{2^4}{3^5}$

(h) Similar to the unbiased case, we will first find the probability of the complement event, which is having no heads. The probability of this is $\frac{1}{3^5}$, which makes our final result $1 - \frac{1}{3^5}$

(i) In this case, since we are working in a nonuniform probability space (getting 4 heads and 3 heads don't have the same probability), we need to separately consider the events with different numbers of heads to find our result. This will get us:

$$\binom{5}{3}\frac{2^3}{3^5} + \binom{5}{4}\frac{2^4}{3^5} + \binom{5}{5}\frac{2^5}{3^5}$$

2 Weathermen

Tom is a weatherman in New York. On days when it snows, Tom correctly predicts the snow 70% of the time. When it doesn't snow, he correctly predicts no snow 95% of the time. In New York, it snows on 10% of all days.

(a) If Tom says that it is going to snow, what is the probability it will actually snow?

(b) What is Tom's overall accuracy?

(c) Tom's friend Jerry is a weatherman in Alaska. Jerry claims that she is a better weatherman than Tom even though her overall accuracy is lower. After looking at their records, you determine that Jerry is indeed better than Tom at predicting snow on snowy days and sun on sunny day. Give an instance of the situation described above.

Hint: what is the weather like in Alaska?

Solution:

(a) Let S be the event that it snows and T be the event that Tom predicts snow.

$$\begin{aligned} P(S|T) &= \frac{P(S\cap T)}{P(T)} \\ &= \frac{P(S\cap T)}{P(S\cap T)+P(\overline{S}\cap T)} \\ &= \frac{.1\times .7}{.1\times .7+.9\times .05} \end{aligned}$$

(b)

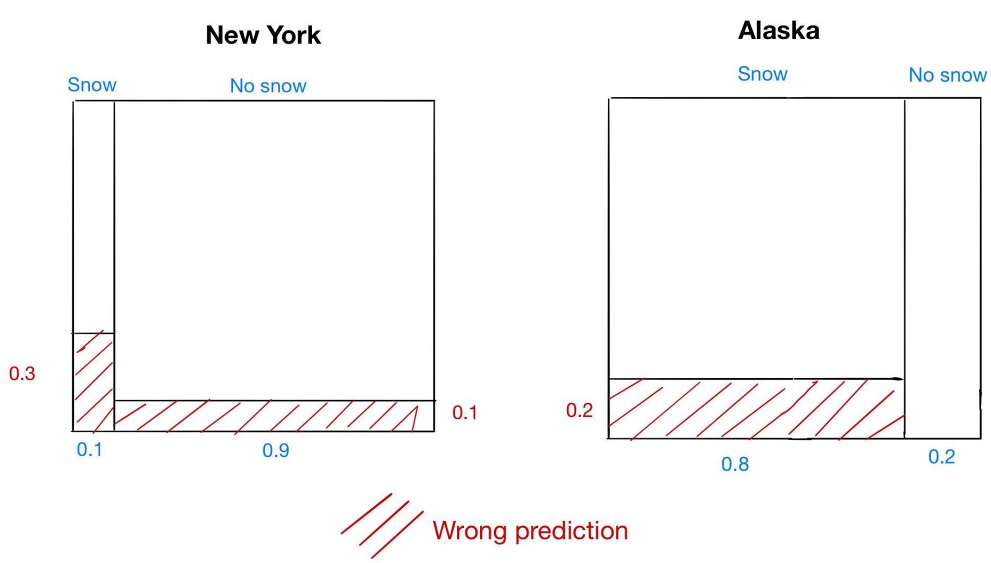
$$\begin{aligned} P(\text{Tom is correct}) &= P(S\cap T)+P(\overline{S}\cap \overline{T}) \\ &= .1\times .7+.9\times .95 \end{aligned}$$

(c) Even though Jerry's overall accuracy is lower, it is still possible that she is a better weatherman if the weather is different.

For example, let's assume that it snows 80% of days in Alaska.

- When it snows, Jerry correctly predicts snow 80% of the time.
- When it doesn't snow, Jerry correctly predicts no snow 100% of the time.

Jerry's overall accuracy turns out to be less than Tom's even though she is better at predicting both categories! The following diagram gives an illustration of the situation. The intuition is that Jerry's error gets penalized more heavily than Tom because it snows more often in Alaska.



For more info on this kind of phenomena, check out Simpson's Paradox!

3 Faulty Lightbulbs

Box 1 contains 1000 lightbulbs of which 10% are defective. Box 2 contains 2000 lightbulbs of which 5% are defective.

(a) Suppose a box is given to you at random and you randomly select a lightbulb from the box. If that lightbulb is defective, what is the probability you chose Box 1?

(b) Suppose now that a box is given to you at random and you randomly select two light- bulbs from the box. If both lightbulbs are defective, what is the probability that you chose from Box 1?

Solution:

(a) Let:

- D denote the event that the lightbulb we selected is defective.
- B_i denote the event that the lightbulb we selected is from Box i.

We wish to compute $\mathbb{P}[B_1 \mid D]$. Using Bayes' Rule we get:

$$\begin{aligned}\mathbb{P}[B_1 \mid D] &= \frac{\mathbb{P}[D \mid B_1]\cdot\mathbb{P}[B_1]}{\mathbb{P}[B_1]\cdot\mathbb{P}[D \mid B_1]+\mathbb{P}[B_2]\cdot\mathbb{P}[D \mid B_2]} \\ &= \frac{0.1\cdot 0.5}{0.5\cdot 0.1+0.5\cdot 0.05} \\ &= \frac{2}{3}\end{aligned}$$

(b) Let:

- D' denote the event that both the lightbulbs we selected are defective.
- B_i denote the event that the lightbulb we selected is from Box i.

We wish to compute $\mathbb{P}[B_1 \mid D']$. Using Bayes' Rule we get:

$$\begin{aligned}\mathbb{P}[B_1 \mid D'] &= \frac{\mathbb{P}[D' \mid B_1]\cdot\mathbb{P}[B_1]}{\mathbb{P}[B_1]\cdot\mathbb{P}[D' \mid B_1]+\mathbb{P}[B_2]\cdot\mathbb{P}[D' \mid B_2]} \\ &= \frac{\frac{100}{1000}\cdot\frac{99}{999}\cdot 0.5}{0.5\cdot\frac{100}{1000}\cdot\frac{99}{999}+0.5\cdot\frac{100}{2000}\cdot\frac{99}{1999}} \\ &= 0.8\end{aligned}$$

4 Solve the Rainbow

Your roommate was having Skittles for lunch and they offer you some. There are five different colors in a bag of Skittles: red, orange, yellow, green, and purple, and there are 20 of each color. You know your roommate is a huge fan of the green Skittles. With probability $1/2$ they ate all of the green ones, with probability $1/4$ they ate half of them, and with probability $1/4$ they only ate 5 green ones.

(a) If you take a Skittle from the bag, what is the probability that it is green?

(b) If you take two Skittles from the bag, what is the probability that at least one is green?

(c) If you take three Skittles from the bag, what is the probability that they are all green?

(d) If all three Skittles you took from the bag are green, what are the probabilities that your roommate had all of the green ones, half of the green ones, or only 5 green ones?

(e) If you take three Skittles from the bag, what is the probability that they are all the same color?

Solution:

(a) We will use the law of total probability. Let G be the event that you take a green Skittles from the bag, A be the event that your roommate ate all of the green Skittles, H be the event that your roommate ate half the green Skittles, and F be the event that your roommate ate five green Skittles. Then, we get the total probability as following:

$$\begin{aligned}
\mathbb{P}(G) &= \mathbb{P}(G \cap A) + \mathbb{P}(G \cap H) + \mathbb{P}(G \cap F) && (1)\\
&= \mathbb{P}(G \mid A)\mathbb{P}(A) + \mathbb{P}(G \mid H)\mathbb{P}(H) + \mathbb{P}(G \mid F)\mathbb{P}(F) && (2)\\
&= 0 \cdot \frac{1}{2} + \frac{10}{90} \cdot \frac{1}{4} + \frac{15}{95} \cdot \frac{1}{4} \approx 0.0673 && (3)
\end{aligned}$$

(b) We will consider the complement event, that neither of them are green. Let's call the event that at least one of them is green B, this makes the complement $\overline{B}$ the event that neither Skittles are green. Using the same approach as the previous part, we will get the following:

$$\begin{aligned}
\mathbb{P}(\overline{B}) &= \mathbb{P}(\overline{B} \cap A) + \mathbb{P}(\overline{B} \cap H) + \mathbb{P}(\overline{B} \cap F) && (4)\\
&= \mathbb{P}(\overline{B} \mid A)\mathbb{P}(A) + \mathbb{P}(\overline{B} \mid H)\mathbb{P}(H) + \mathbb{P}(\overline{B} \mid F)\mathbb{P}(F) && (5)\\
&= 1 \cdot \frac{1}{2} + \frac{80}{90} \cdot \frac{79}{89} \cdot \frac{1}{4} + \frac{80}{95} \cdot \frac{79}{94} \cdot \frac{1}{4} \approx 0.874 && (6)
\end{aligned}$$

This makes our final answer the following:

$$\mathbb{P}(B) = 1 - \mathbb{P}(\overline{B}) \approx 0.126$$

(c) Let's call the event of having 3 green Skittles G_3. This event is impossible if our roommate ate all the green Skittles.

If they ate half, we have the probability of G_3 as

$$\frac{10\times 9\times 8}{90\times 89\times 88}.$$

We can see this by noticing that given our roommate ate half the green Skittles, there will be 10 green Skittles left out of the 90 that are still in the bag. After the first one is removed, there will be 9 out of 89 that are green, and so on.

Similarly, if they ate only five green Skittles, we have the probability of G_3 as

$$\frac{15\times 14\times 13}{95\times 94\times 93},$$

giving us the final result as:

$$\begin{aligned}\mathbb{P}(G_3) &= \mathbb{P}(G_3 \mid A)\mathbb{P}(A)+\mathbb{P}(G_3 \mid H)\mathbb{P}(H)+\mathbb{P}(G_3 \mid F)\mathbb{P}(F) && (7)\\ &= 0\cdot\frac{1}{2}+\frac{10\times 9\times 8}{90\times 89\times 88}\cdot\frac{1}{4}+\frac{15\times 14\times 13}{95\times 94\times 93}\cdot\frac{1}{4} && (8)\\ &\approx 0.00108 && (9)\end{aligned}$$

(d) We can use the Bayes Rule to solve this.

$$\mathbb{P}(A \mid G_3)=\frac{\mathbb{P}(G_3\cap A)}{\mathbb{P}(G_3)}=\frac{\mathbb{P}(G_3 \mid A)\mathbb{P}(A)}{\mathbb{P}(G_3)}=\frac{0\times 1/2}{0.00108}=0$$

This makes intuitive sense, since if you took three green Skittles out of the bag, it is impossible that your roommate ate all of them. Using it for the two other conditions, we get:

$$\mathbb{P}(H \mid G_3)=\frac{\mathbb{P}(G_3\cap H)}{\mathbb{P}(G_3)}=\frac{\mathbb{P}(G_3 \mid H)\mathbb{P}(H)}{\mathbb{P}(G_3)}=\frac{10\times 9\times 8}{90\times 89\times 88}\cdot\frac{1}{4}\cdot\frac{1}{0.00108}\approx 0.237$$

$$\mathbb{P}(F \mid G_3)=\frac{\mathbb{P}(G_3\cap F)}{\mathbb{P}(G_3)}=\frac{\mathbb{P}(G_3 \mid F)\mathbb{P}(F)}{\mathbb{P}(G_3)}=\frac{15\times 14\times 13}{95\times 94\times 93}\cdot\frac{1}{4}\cdot\frac{1}{0.00108}\approx 0.763$$

Note that the sum of these probabilities add up to 1.

(e) We can divide this into two cases. If the color of all the Skittles is green, we have already calculated the probability in the previous part.

For all other colors, we can notice that the probabilities will have the same structure, and since these are disjoint events, we can add them to get our final result. Let's find the probability for the case of getting three red Skittles, let's call this event R_3. We find this probability as follows:

$$\begin{aligned}\mathbb{P}(R_3) &= \mathbb{P}(R_3 \mid A)\mathbb{P}(A)+\mathbb{P}(R_3 \mid H)\mathbb{P}(H)+\mathbb{P}(R_3 \mid F)\mathbb{P}(F) && (10)\\ &= \frac{20\times 19\times 18}{80\times 79\times 78}\cdot\frac{1}{2}+\frac{20\times 19\times 18}{90\times 89\times 88}\cdot\frac{1}{4}+\frac{20\times 19\times 18}{95\times 94\times 93}\cdot\frac{1}{4} && (11)\\ &\approx 0.0114 && (12)\end{aligned}$$

If we call the probability of getting three Skittles of the same color X_3, we can find it by adding the probability for the events for different colors such as G_3, and R_3. The probability for getting 3 of the same color for yellow, orange, and purple will be the same as it was for red. Using the same name convention for red and green for the other colors, this can be summed up as:

$$\begin{aligned} \mathbb{P}(X_3) &= \mathbb{P}(G_3) + \mathbb{P}(R_3) + \mathbb{P}(Y_3) + \mathbb{P}(O_3) + \mathbb{P}(P_3) && (13) \\ &= \mathbb{P}(G_3) + 4\mathbb{P}(R_3) && (14) \end{aligned}$$

The above holds since these are all disjoint events, we can't get all three Skittles to be the same color for different colors at the same time. Overall, getting this means we are adding these probabilities, giving us:

$$\mathbb{P}(X_3) = \mathbb{P}(G_3) + 4 \cdot \mathbb{P}(R_3) \approx 0.0468$$

5 Playing Strategically

Bob, Eve and Carol bought new slingshots. Bob is not very accurate hitting his target with probability $1/3$. Eve is better, hitting her target with probability $2/3$. Carol never misses. They decide to play the following game: They take turns shooting each other. For the game to be fair, Bob starts first, then Eve and finally Carol. Any player who gets shot has to leave the game. The last person standing wins the game. What is Bob's best course of action regarding his first shot?

(a) Compute the probability of the event E_1 that Bob wins in a duel against Eve alone, assuming he shoots first. (Hint: Let x be the probability Bob wins in a dual against Eve alone, assuming he fires first. If Bob misses his first shot and then Eve misses her first shot, what is the probability Bob wins in terms of x?)

(b) Compute the probability of the event E_2 that Bob wins in a duel against Eve alone, assuming he shoots second.

(c) Compute the probability of the same events for a duel of Bob against Carol.

(d) Assuming that both Eve and Carol play rationally, conclude that Bob's best course of action is to shoot into the air (i.e., intentionally miss)! (Hint: What happens if Bob misses? What if he doesn't?)

Solution:

(a) Compute the probability of the event E_1 that Bob wins in a duel against Eve alone, assuming he shoots first.

Observe that:

$$\begin{aligned} \mathbb{P}[E_1] &= \mathbb{P}[\text{Bob hits Eve }] + \mathbb{P}[\text{Bob misses Eve}]\mathbb{P}[\text{Eve misses Bob}]\mathbb{P}[E_1] \\ &= \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{3}\mathbb{P}[E_1] \end{aligned}$$

Thus, $\mathbb{P}[E_1] = \frac{3}{7}$.

(b) Compute the probability of the event E_2 that Bob wins in a duel against Eve alone, assuming he shoots second.

Observe that:

$$\begin{aligned}\mathbb{P}[E_2] &= \mathbb{P}[\text{Eve misses Bob}]\left(\mathbb{P}[\text{Bob hits Eve}] + \mathbb{P}[\text{Bob misses Eve}]\mathbb{P}[E_2]\right)\\ &= \frac{1}{3}\left(\frac{1}{3} + \frac{2}{3}\mathbb{P}[E_2]\right)\end{aligned}$$

Thus, $\mathbb{P}[E_2] = \frac{1}{7}$.

(c) Compute the probability of the same events for a duel of Bob against Carol alone.

The probability of the event E_3 that Bob, with shot, survives against Carol is:

$$\begin{aligned}\mathbb{P}[E_3] &= \mathbb{P}[\text{Bob hits Carol}] + \mathbb{P}[\text{Bob misses Carol}]\mathbb{P}[\text{Carol misses Bob}]\mathbb{P}[E_3]\\ &= \frac{1}{3} + \frac{2}{3}\cdot 0\\ &= \frac{1}{3}\ .\end{aligned}$$

The probability of the event E_4 that Bob, without shot, survives against Carol is:

$$\mathbb{P}[E_4] \leq \mathbb{P}[\text{Carol misses}] = 0\ .$$

(d) To maximize their chances each player prefers to be left with a weaker opponent. This means that Eve would not shoot at Bob in preference to Carol, and Carol will not shoot at Bob in preference to Eve. Therefore if Bob misses, he will not be shot at until either Eve or Carol lose and he will either be left standing with Eve or Carol, with or without the shot.

So Bob is best off not shooting anyone since the advantage he gains by having the first shot exceeds any possible benefit of facing Eve rather than Carol. He should shoot into the air.

6 Minesweeper

Minesweeper is a game that takes place on a grid of squares. When you click a square, it disappears to reveal either an integer $\in [1,8]$, a mine, or a blank space. If it reveals a mine, you instantly lose. If it reveals a number, that number refers to the number of mines adjacent to that square (including diagonally adjacent). If it reveals a blank space, there were 0 mines adjacent to it.

You are playing on a 8x8 board with 10 mines randomly distributed across the board. In your first move, you click a square near the center of the board.

(a) What is the probability that the square reveals...

i. a mine?

ii. a blank space?

iii. the number k?

(b) The first square you picked revealed the number k. For your next move, you want to minimize the probability of picking a mine. Should you pick a square adjacent to your first pick, or a different square? Your answer should depend on the value of k.

(c) Your first move resulted in the number 1. You pick the square to the right for your next move. What is the probability that this square reveals the number 4?

Solution:

(a) i. There are 10 mines and 64 squares, so the probability of a square being a mine is $\frac{10}{64}$

ii. This is the probability that the picked square and its 8 adjacent squares are not mines. Then, we calculate the probability that all 10 mines are among the other 55 squares. $\frac{\binom{55}{10}}{\binom{64}{10}}$

iii. $\frac{\binom{8}{k}\binom{55}{10-k}}{\binom{64}{10}}$. We choose locations for the k adjacent mines and locations for the remaining $10-k$ mines. The denominator is the total number of possible arrangements of mines.

(b) The probability of picking a mine if you click an adjacent square is $\frac{k}{8}$. The probability of picking a mine if you click a different square is $\frac{10-k}{55}$. You should pick an adjacent square if $\frac{k}{8} \leq \frac{10-k}{55}$. This occurs only when $k = 1$.

(c) The square to the right will share 4 neighbors with the original square. In order to reveal the number 4, one of the mutual neighbors must be a mine. The three new neighbors must also be mines.

The probability that one of the mutual neighbors is a mine is $\frac{1}{2}$. Given that one of the mutual neighbors is a mine, the probability that the three new neighbors are also all mines is $\frac{\binom{52}{6}}{\binom{55}{9}}$ The probability that both these events occur must then be:

$$\frac{1}{2} \times \frac{\binom{52}{6}}{\binom{55}{9}}$$