CS 70 Discrete Mathematics and Probability Theory
Spring 2019 Babak Ayazifar and Satish Rao

HW 9

Due: Friday, April 5, 2019 at 10 PM

Sundry

Before you start your homework, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Cliques in Random Graphs

Consider a graph $G = (V, E)$ on n vertices which is generated by the following random process: for each pair of vertices u and v, we flip a fair coin and place an (undirected) edge between u and v if and only if the coin comes up heads. So for example if $n = 2$, then with probability $1/2$, $G = (V, E)$ is the graph consisting of two vertices connected by an edge, and with probability $1/2$ it is the graph consisting of two isolated vertices.

(a) What is the size of the sample space?

(b) A k-clique in graph is a set of k vertices which are pairwise adjacent (every pair of vertices is connected by an edge). For example a 3-clique is a triangle. What is the probability that a particular set of k vertices forms a k-clique?

(c) Prove that $\binom{n}{k} \leq n^k$.

Optional: Can you come up with a combinatorial proof? Of course, an algebraic proof would also get full credit.

(d) Prove that the probability that the graph contains a k-clique, for $k \geq 4\log n + 1$, is at most $1/n$. (The log is taken base 2).

Hint: Apply the union bound and part (c).

2 Identity Theft

A group of n friends go to the gym together, and while they are playing basketball, they leave their bags against the nearby wall. An evildoer comes, takes the student ID cards from the bags, randomly rearranges them, and places them back in the bags, one ID card per bag.

(a) What is the probability that no one receives his or her own ID card back?

Hint: Use the inclusion-exclusion principle.

(b) What is the limit of this proability as $n \to \infty$?

Hint: $\mathrm{e}^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

3 Balls and Bins, All Day Every Day

You throw n balls into n bins uniformly at random, where n is a positive *even* integer.

(a) What is the probability that exactly k balls land in the first bin, where k is an integer $0 \le k \le n$?

(b) What is the probability p that at least half of the balls land in the first bin? (You may leave your answer as a summation.)

(c) Using the union bound, give a simple upper bound, in terms of p, on the probability that some bin contains at least half of the balls.

(d) What is the probability, in terms of p, that at least half of the balls land in the first bin, or at least half of the balls land in the second bin?

(e) After you throw the balls into the bins, you walk over to the bin which contains the first ball you threw, and you randomly pick a ball from this bin. What is the probability that you pick up the first ball you threw? (Again, leave your answer as a summation.)

4 Babak's Dice

Professor Ayazifar rolls three fair six-sided dice.

(a) Let X denote the maximum of the three values rolled. What is the distribution of X (that is, $\mathbb{P}[X = x]$ for $x = 1,2,3,4,6$)? You can leave your final answer in terms of "x". [*Hint*: Try to first compute $\mathbb{P}[X \le x]$ for $x = 1,2,3,4,5,6$].

(b) Let Y denote the minimum of the three values rolled. What is the distribution of Y?

5 Testing Model Planes

Dennis is testing model airplanes. He starts with n model planes which each independently have probability p of flying successfully each time they are flown, where $0 < p < 1$. Each day, he flies every single plane and keeps the ones that fly successfully (i.e. don't crash), throwing away all other models. He repeats this process for many days, where each "day" consists of Dennis flying any remaining model planes and throwing away any that crash. Let X_i be the random variable representing how many model planes remain after i days. Note that $X_0 = n$. Justify your answers for each part.

(a) What is the distribution of X_1? That is, what is $\mathbb{P}[X_1 = k]$?

(b) What is the distribution of X_2? That is, what is $\mathbb{P}[X_2 = k]$? Show that X_2 follows a binomial distribution by finding some n' and p' such that $X_2 \sim \text{Binom}(n', p')$.

(c) Repeat the previous part for X_t for arbitrary $t \geq 1$.

(d) What is the probability that at least one model plane still remains (has not crashed yet) after t days? Do not have any summations in your answer.

(e) Considering only the first day of flights, is the event A_1 that the first and second model planes crash independent from the event B_1 that the second and third model planes crash? Recall that two events A and B are independent if $\mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B]$. Prove your answer using this definition.

(f) Considering only the first day of flights, let A_2 be the event that the first model plane crashes *and* exactly two model planes crash in total. Let B_2 be the event that the second plane crashes on the first day. What must n be equal to in terms of p such that A_2 is independent from B_2? Prove your answer using the definition of independence stated in the previous part.

(g) Are the random variables X_i and X_j, where $i < j$, independent? Recall that two random variables X and Y are independent if $\mathbb{P}[X = k_1 \cap Y = k_2] = \mathbb{P}[X = k_1]\mathbb{P}[Y = k_2]$ for all k_1 and k_2. Prove your answer using this definition.

6 Cookie Jars

You have two jars of cookies, each of which starts with n cookies initially. Every day, when you come home, you pick one of the two jars randomly (each jar is chosen with probability $1/2$) and eat one cookie from that jar. One day, you come home and reach inside one of the jars of cookies, but you find that is empty! Let X be the random variable representing the number of remaining cookies in non-empty jar at that time. What is the distribution of X?