1 A Meeting of Three

Xavier, Yvette, and Zachary each arrive at the movie theater at times uniformly distributed in the interval (3:00, 4:00). Their arrival times are independent. Each person waits $\frac{1}{4}$ hour after their arrival before heading into the theater. What is the probability they all see each other before going into the theater?

Solution:

Denote Xavier, Yvette, and Zachary's arrival times as X, Y, and Z respectively. Note that they are i.i.d. $U[0,1]$. (Here 1 represents 1 hour).

Solution 1

Consider the region where $X > Y$. For X, Y, and Z to all be within the same 1/4 length interval, Y can be between $X - 1/4$ and X. However since X, Y cannot take negative values, for $X \in [0, 1/4]$, Y can range from 0 to X. Now, for Z to be within 1/4 of the other two, it can go from $X - 1/4$ to $Y + 1/4$ since $X > Y$. However, again, since X, Y, Z must take values in $[0,1]$, for $X \in [0, 1/4]$, Z goes from 0 to $Y + 1/4$, and for $Y \in [3/4, 1]$, Z goes from $X - 1/4$ to 1. By symmetry, the case $X < Y$ has the same probability, so we multiply the result of the $X > Y$ case by 2.Using the joint density $f_{X,Y,Z} = 1$, $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$, we compute the following integrals:

$$\begin{aligned}
\mathbb{P}(\text{meeting}) &= \mathbb{P}(\max(X,Y,Z) - \min(X,Y,Z) < 1/4) \\
&= 2 \cdot \mathbb{P}(X > Y \cap \max(X,Z) - Y < 1/4) \\
&= 2 \cdot \Big(\int_0^{1/4} \int_0^x \int_0^{y+1/4} \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x + \int_{1/4}^{3/4} \int_{x-1/4}^{x} \int_{x-1/4}^{y+1/4} \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\
&\qquad + \int_{3/4}^{1} \int_{x-1/4}^{3/4} \int_{x-1/4}^{y+1/4} \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x + \int_{3/4}^{1} \int_{3/4}^{x} \int_{x-1/4}^{1} \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \Big) \\
&= 2 \cdot \left(\frac{1}{96} + \frac{3}{64} + \frac{1}{96} + \frac{1}{96} \right) \\
&= \frac{5}{32}.
\end{aligned}$$

Solution 2 Consider that there are six equally possible arrival orders. By symmetry, the total probability in question is 6 times that of the case where $X < Y < Z$. Then,

$$
\begin{aligned}
P(\text{Meeting}) &= 6\,[\int_{x=0}^{3/4}\int_{y=x}^{x+1/4}\int_{z=y}^{x+1/4} dz\,dy\,dx \\
&+ \int_{x=3/4}^{1}\int_{y=x}^{1}\int_{z=y}^{1} dz\,dy\,dx] \\
&= 6\,[\frac{3}{128}+\frac{1}{3\times 128}] \\
&= \frac{5}{32}
\end{aligned}
$$

2 Exponential Practice

(a) Let $X_1, X_2 \sim \text{Exponential}(\lambda)$ be independent, $\lambda > 0$. Calculate the density of $Y := X_1 + X_2$. [*Hint*: One way to approach this problem would be to compute the CDF of Y and then differentiate the CDF.]

(b) Let $t > 0$. What is the density of X_1, conditioned on $X_1 + X_2 = t$? [*Hint*: Once again, it may be helpful to consider the CDF $\mathbb{P}(X_1 \le x \mid X_1 + X_2 = t)$. To tackle the conditioning part, try conditioning instead on the event $\{X_1 + X_2 \in [t, t+\varepsilon]\}$, where $\varepsilon > 0$ is small.]

Solution:

(a) Let $y > 0$. Observe that if $X_1 + X_2 \le y$, then since $X_1, X_2 \ge 0$, it follows that $X_1 \le y$ and $X_2 \le y - X_1$.

$$
\begin{aligned}
\mathbb{P}(Y \le y) = \mathbb{P}(X_1 \le y, X_2 \le y - X_1) &= \int_0^y \int_0^{y-x_1} \lambda \exp(-\lambda x_1)\lambda \exp(-\lambda x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1 \\
&= \lambda^2 \int_0^y \exp(-\lambda x_1)\cdot \frac{1-\exp(-\lambda(y-x_1))}{\lambda}\,\mathrm{d}x_1 \\
&= \lambda \int_0^y \big(\exp(-\lambda x_1) - \exp(-\lambda y)\big)\,\mathrm{d}x_1 = \lambda\Big(\frac{1-\exp(-\lambda y)}{\lambda} - y\exp(-\lambda y)\Big).
\end{aligned}
$$

Upon differentiating the CDF, we have

$$
\begin{aligned}
f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y}\mathbb{P}(Y \le y) &= \lambda \exp(-\lambda y) - \lambda \exp(-\lambda y) + \lambda^2 y \exp(-\lambda y) \\
&= \lambda^2 y \exp(-\lambda y), \qquad y > 0.
\end{aligned}
$$

Alternative solution: Since X_1 and X_2 are limits of X_1^n/n and X_2^n/n, where X_1^n and X_2^n are independent $\text{Geom}(p_n = \lambda/n)$, we know that $f_Y(y)\mathrm{d}y = \lim_{n\to\infty}\mathbb{P}\left[(X_1^n + X_2^n)/n = y\right]$, i.e. $f(y) = \lim_{n\to\infty} n\mathbb{P}\left[X_1^n + X_2^n = ny\right]$. But from worksheet 11b we know that

$$
n\mathbb{P}\left[X_1^n + X_2^n = ny\right] = n(ny-1)(1-p_n)^{ny-2}p_n^2 = \lambda^2\left(y - \frac{1}{n}\right)\left(1 - \frac{\lambda}{n}\right)^{ny-2},
$$

which as $n \to \infty$ converges to $\lambda^2 y e^{-\lambda y}$ as desired.

(b) Let $0 \le x \le t$. Following the hint, we have

$$\begin{aligned}
\mathbb{P}(X_1 \le x \mid X_1 + X_2 \in [t, t+\varepsilon]) &= \frac{\mathbb{P}(X_1 \le x, X_1 + X_2 \in [t, t+\varepsilon])}{\mathbb{P}(X_1 + X_2 \in [t, t+\varepsilon])} \\
&= \frac{\mathbb{P}(X_1 \le x, X_2 \in [t - X_1, t - X_1 + \varepsilon])}{f_Y(t) \cdot \varepsilon} \\
&= \frac{\int_0^x \int_{t-x_1}^{t-x_1+\varepsilon} \lambda \exp(-\lambda x_1) \lambda \exp(-\lambda x_2) \, \mathrm{d}x_2 \, \mathrm{d}x_1}{\lambda^2 t \exp(-\lambda t) \cdot \varepsilon} \\
&= \frac{\lambda^2 \int_0^x \exp(-\lambda x_1) \exp(-\lambda (t - x_1)) \varepsilon \, \mathrm{d}x_1}{\lambda^2 t \exp(-\lambda t) \cdot \varepsilon} = \frac{\int_0^x \mathrm{d}x_1}{t} = \frac{x}{t}.
\end{aligned}$$

This means that the density is

$$f_{X_1 \mid X_1 + X_2}(x \mid t) = \frac{\mathrm{d}}{\mathrm{d}x} \mathbb{P}(X \le x \mid X_1 + X_2 = t) = \frac{1}{t}, \qquad x \in [0, t],$$

which means that conditioned on $X_1 + X_2 = t$, X_1 is actually uniform on the interval $[0, t]$!

Alternative solution: Using the discrete approximations X_1^n/n and X_2^n/n as in the alternative solution to part (a), we have

$$\begin{aligned}
n \cdot \mathbb{P}(X_1^n = xn \mid X_1^n + X_2^n = tn) &= n \frac{\mathbb{P}(X_1^n = xn \cap X_2^n = tn - xn)}{\mathbb{P}(X_1^n + X_2^n = tn)} = n \frac{(1-p_n)^{xn-1} p_n (1-p_n)^{tn-xn-1} p_n}{(tn-1)(1-p_n)^{tn-2} p_n^2} \\
&= \frac{1}{t - 1/n},
\end{aligned}$$

which converges to $1/t$ as $n \to \infty$ just like before.

3 Normal Darts?

Alex and John are playing a game of darts. Let (X_a, Y_a) and (X_j, Y_j) denote the coordinates of Alex's and John's darts on the board and are distributed in the following way:

1. $X_a, Y_a \sim \mathbb{N}(0, 1)$ independently
2. X_j, Y_j are distributed uniformly in a circle of radius 3

The winner of the game is determined by whoever's darts is closer to the center of the board at $(0, 0)$. In this question, we will compute the probability that Alex wins the game. We will denote the squared distances of the darts from the center by $r_a = X_a^2 + Y_a^2$ and by $r_j = X_j^2 + Y_j^2$.

(a) What is the distribution of r_a?
Hint: Consider the joint distribution and the following change of variables formula: Suppose we want to integrate the function $f(x,y)$ over the circle $\left(\sqrt{x^2+y^2} \leq R\right)$. Then, we have the following change of variables formula:

$$\int_{\sqrt{x^2+y^2}\leq R} f(x,y)dxdy = \int_0^R \int_0^{2\pi} f(r\cos\theta, r\sin\theta) r d\theta dr$$

You may find the identity $\sin(\theta)^2 + \cos(\theta)^2 = 1$ useful.

(b) What is the distribution of r_j? (Hint: Try computing the CDF first)

(c) What is the probability that Alex wins the game?

Solution:

(a) Since X_a, X_j are independent, the joint distribution of (X_a, X_j) is given by:

$$f(x,y) = f(x)f(y) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2})\frac{1}{\sqrt{2\pi}}\exp(-\frac{y^2}{2}) = \frac{1}{2\pi}\exp\left(-\frac{x^2+y^2}{2}\right)$$

Now, we have:

$$\begin{aligned}
\mathbb{P}(r_a \leq t) &= \int_{x^2+y^2\leq t} \frac{1}{2\pi}\exp\left(-\frac{x^2+y^2}{2}\right)dxdy = \int_{\sqrt{x^2+y^2}\leq\sqrt{t}} \frac{1}{2\pi}\exp\left(-\frac{x^2+y^2}{2}\right)dxdy \\
&= \int_{r=1}^{\sqrt{t}}\int_{\theta=0}^{2\pi} \frac{1}{2\pi}\exp\left(-\frac{r^2}{2}\right) r d\theta dr = \int_{r=0}^{\sqrt{t}} r\exp(-r^2/2)dr \\
&= -\exp(-r^2/2)|_{r=0}^{\sqrt{t}} = 1-\exp(-t/2)
\end{aligned}$$

(b) Since X_j, Y_j are distributed uniformly in the circle, the cdf of r_j is the area of the inner circle of radius k over the total area of the circle. Therefore, the CDF of the distribution of r_j is given by:

$$\mathbb{P}\left[r_j \leq k\right] = \frac{\pi \cdot k}{\pi \cdot 3^2} = \frac{k}{9}$$

Therefore, we get that the pdf of r_j is given by:

$$f_{r_j}(x) = \frac{1}{9}I\{x \leq 9\}$$

(c) We want to calculate the probability of Alex winning. We integrate over the possible values that John can be away from the board (r_j can be from 0 to 9):

$$\mathbb{P}[\text{Alex Winning}] = \mathbb{P}[r_a \leq r_j] = \int_{t=0}^{9} \mathbb{P}[r_a \leq t] f_{r_j}(t)dt = \int_{t=0}^{9} \frac{1}{9}(1-e^{-\frac{t}{2}}) = \frac{7}{9} + \frac{2}{9}e^{-\frac{9}{2}}$$

4 Why Is It Gaussian?

Let X be a normally distributed random variable with mean μ and variance σ^2. Let $Y = aX + b$, where a and b are non-zero real numbers. Show explicitly that Y is normally distributed with mean $a\mu + b$ and variance $a^2\sigma^2$. The PDF for the Gaussian Distribution is $\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$.

Solution:

Problem and solution taken from *A First Course in Probability* by Sheldon Ross, 8th edition.

Let $a > 0$.

We start with the cumulative distribution function (CDF) of Y, F_Y.

$$\begin{aligned}
F_Y(x) &= \mathbb{P}[Y \leq x] && \text{By definition of CDF} \\
&= \mathbb{P}[aX + b \leq x] && \text{Plug in } Y = aX + b \\
&= \mathbb{P}\left[X \leq \frac{x-b}{a}\right] && \text{Because } a > 0 \\
&= F_X\left(\frac{x-b}{a}\right) && \text{By definition of CDF. } F_X \text{ denotes the CDF of } X.
\end{aligned} \tag{1}$$

Let f_Y denote the probability density function (PDF) of Y.

$$\begin{aligned}
f_Y(x) &= \frac{\mathrm{d}}{\mathrm{d}x} F_Y(x) && \text{The PDF is the derivative of the CDF.} \\
&= \frac{\mathrm{d}}{\mathrm{d}x} F_X\left(\frac{x-b}{a}\right) && \text{Plug in the result from (1)} \\
&= \frac{1}{a} \cdot f_X\left(\frac{x-b}{a}\right) && \begin{array}{l}\text{PDF is the derivative of CDF.} \\ \text{Apply chain rule, } \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x-b}{a}\right) = \frac{1}{a}.\end{array} \\
&= \frac{1}{a} \cdot \frac{1}{\sigma\sqrt{2\pi}} \cdot \mathrm{e}^{-((x-b)/a-\mu)^2/(2\sigma^2)} && X \sim \mathcal{N}(\mu, \sigma^2). \\
&= \frac{1}{a\sigma\sqrt{2\pi}} \cdot \mathrm{e}^{-(x-b-a\mu)^2/(2\sigma^2 a^2)} && \frac{x-b}{a} - \mu = \frac{1}{a}(x - b - a\mu)
\end{aligned} \tag{2}$$

We have shown that f_Y equals the probability density function of a normal random variable with mean $b + a\mu$ and variance $\sigma^2 a^2$. So, Y is normally distributed with mean $b + a\mu$ and variance $\sigma^2 a^2$. The proof is done for $a > 0$. The proof for $a < 0$ is similar.

5 Moments of the Gaussian

$\mathbb{E}[X^k]$, where $k \in \mathbb{N}$, is called the *kth moment* of the distribution. In this problem, we will calculate the moments of a standard normal distribution.

(a) Prove the identity

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{tx^2}{2}\right) dx = t^{-1/2} \tag{3}$$

for $t > 0$.

Hint: Consider a normal distribution with variance $\frac{1}{t}$ and mean 0.

(b) For the rest of the problem, X is a standard normal distribution (with mean 0 and variance 1). Use part a to compute $\mathbb{E}[X^{2k}]$ for $k \in \mathbb{N}$. [*Hint*: Try differentiating with respect to t k times.]

(c) Compute $\mathbb{E}[X^{2k+1}]$ for $k \in \mathbb{N}$.

Solution:

(a) Note that a normal distribution with mean 0 and variance t^{-1} has the density function

$$f(x) = \frac{\sqrt{t}}{\sqrt{2\pi}} \exp\left(-\frac{tx^2}{2}\right),$$

and since the density must integrate to 1, we see that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{tx^2}{2}\right) \mathrm{d}x = t^{-1/2}.$$

(b) Differentiating k times with respect to t, we obtain

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-1)^k \frac{x^{2k}}{2^k} \exp\left(-\frac{tx^2}{2}\right) \mathrm{d}x = (-1)^k \frac{1 \cdot 3 \cdots (2k-3) \cdot (2k-1)}{2^k} t^{-(2k+1)/2},$$

and upon setting $t = 1$, we have

$$\mathbb{E}[X^{2k}] = \int_{-\infty}^{\infty} x^{2k} \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x = \prod_{i=1}^{k} (2i-1).$$

This is sometimes denoted $(2k-1)!!$. Note that we can also write the result as

$$\mathbb{E}[X^{2k}] = (2k-1)!! = \frac{(2k)!}{2 \cdot 4 \cdots (2k-2) \cdot (2k)} = \frac{(2k)!}{2^k k!}.$$

Remark: Technically, we would have to justify why we are allowed to differentiate under the integral.

(c) $\mathbb{E}[X^{2k+1}] = 0$, since the density function is symmetric around 0.

6 Noisy Love

Suppose you have confessed to your love interest on Valentine's Day and you are waiting to hear back. Your love interest is trying to send you a binary message: "0" means that your love interest is not interested in you, while "1" means that your love interest reciprocates your feelings. Let X be your love interest's message for you. Your current best guess of X has $\mathbb{P}(X = 0) = 0.7$ and $\mathbb{P}(X = 1) = 0.3$. Unfortunately, your love interest sends you the message through a noisy channel, and instead of receiving the message X, you receive the message $Y = X + \varepsilon$, where ε is independent Gaussian noise with mean 0 and variance 0.49.

(a) First, you decide upon the following rule: if you observe $Y > 0.5$, then you will assume that your love interest loves you back, whereas if you observe $Y \le 0.5$, then you will assume that your love interest is not interested in you. What is the probability that you are correct using this rule? (Express your answer in terms of the CDF of the standard Gaussian distribution $\Phi(z) = \mathbb{P}(\mathcal{N}(0,1) \le z)$, and then evaluate your answer numerically.)

(b) Suppose you observe $Y = 0.6$. What is the probability that your love interest loves you back? [*Hint*: This problem requires conditioning on an event of probability 0, namely, the event $\{Y = 0.6\}$. To tackle this problem, think about conditioning on the event $\{Y \in [0.6, 0.6+\delta]\}$, where $\delta > 0$ is small, so that $f_Y(0.6)\cdot\delta \approx \mathbb{P}(Y \in [0.6, 0.6+\delta])$, and then apply Bayes Rule.]

(c) Suppose you observe $Y = y$. For what values is it more likely than not that your love interest loves you back? [*Hint*: As before, instead of considering $\{Y = y\}$, you can consider the event $\{Y \in [y, y+\delta]\}$ for small $\delta > 0$. So, when is $\mathbb{P}(X = 1 \mid Y \in [y, y+\delta]) \ge \mathbb{P}(X = 0 \mid Y \in [y, y+\delta])$?]

(d) Your new rule is to assume that your love interest loves you back if (based on the value of Y that you observe) it is more likely than not that your love interest loves you back. Under this new rule, what is the probability that you are correct?

Solution:

(a) The probability that you are correct is

$$
\begin{aligned}
&\mathbb{P}(X=0)\mathbb{P}(Y \le 0.5 \mid X=0) + \mathbb{P}(X=1)\mathbb{P}(Y > 0.5 \mid X=1) \\
&\quad = 0.7\mathbb{P}(\mathcal{N}(0,0.49) \le 0.5) + 0.3\mathbb{P}(\mathcal{N}(1,0.49) > 0.5) \\
&\quad = 0.7\mathbb{P}\left(\mathcal{N}(0,1) \le \frac{0.5}{0.7}\right) + 0.3\mathbb{P}\left(\mathcal{N}(0,1) > -\frac{0.5}{0.7}\right) = \mathbb{P}\left(\mathcal{N}(0,1) \le \frac{5}{7}\right) = \Phi\left(\frac{5}{7}\right) \\
&\quad \approx 0.762.
\end{aligned}
$$

(b) By conditioning on $\{Y \in [0.6, 0.6+\delta]\}$, we have

$$
\begin{aligned}
&\mathbb{P}(X = 1 \mid Y \in [0.6, 0.6+\delta]) \\
&\quad = \frac{\mathbb{P}(X=1)\mathbb{P}(Y \in [0.6, 0.6+\delta] \mid X=1)}{\mathbb{P}(X=0)\mathbb{P}(Y \in [0.6, 0.6+\delta] \mid X=0) + \mathbb{P}(Y=1)\mathbb{P}(Y \in [0.6, 0.6+\delta] \mid X=1)} \\
&\quad = \frac{\mathbb{P}(X=1) f_{Y|1}(0.6)}{\mathbb{P}(X=0) f_{Y|0}(0.6) + \mathbb{P}(X=1) f_{Y|1}(0.6)},
\end{aligned}
$$

where $f_{Y|0}$ is the density of a Gaussian with mean 0 and variance 0.49, and $f_{Y|1}$ is the density of a Gaussian with mean 1 and variance 0.49. Although the expression above may look intimidating, *this is just Bayes rule where* $\mathbb{P}(Y = 0.6 \mid X = x)$ *has been replaced with* $f_{Y|x}(0.6)$. The moral of the story is that conditioning in continuous probability seems strange at first, but it is essentially the same as conditioning in discrete probability, with densities taking the place of probability mass functions.

Now, filling in the probabilities, we have

$$\begin{aligned}
&\mathbb{P}(X = 1 \mid Y = 0.6) \\
&\quad = \frac{0.3 \cdot (2\pi)^{-1/2} \exp(-0.4^2/(2 \cdot 0.49))}{0.7 \cdot (2\pi)^{-1/2} \exp(-0.6^2/(2 \cdot 0.49)) + 0.3 \cdot (2\pi)^{-1/2} \exp(-0.4^2/(2 \cdot 0.49))} \approx 0.345.
\end{aligned}$$

See what happened here? Before, you thought $\mathbb{P}(X = 1) = 0.3$. Observing $Y = 0.6$ gives you slightly more evidence in favor of your love interest loving you back, which increases your belief to $\mathbb{P}(X = 1 \mid Y = 0.6) = 0.345$.

(c) We are looking for

$$\mathbb{P}(X = 1 \mid Y \in [y, y+\delta]) \geq \mathbb{P}(X = 0 \mid Y \in [y, y+\delta])$$

which is equivalent to

$$\mathbb{P}(X = 1 \mid Y \in [y, y+\delta]) \geq \frac{1}{2}.$$

Now, we can compute the LHS as in the previous part:

$$\begin{aligned}
\mathbb{P}(X = 1 \mid Y \in [y, y+\delta]) &= \frac{\mathbb{P}(X = 1) f_{Y|1}(y)}{\mathbb{P}(X = 0) f_{Y|0}(y) + \mathbb{P}(X = 1) f_{Y|1}(y)} \\
&= \frac{0.3 \exp(-(1-y)^2/0.98)}{0.7 \exp(-y^2/0.98) + 0.3 \exp(-(1-y)^2/0.98)} \\
&= \frac{1}{1 + (0.7/0.3) \exp(((1-y)^2 - y^2)/0.98)}.
\end{aligned}$$

In order to make the RHS $\geq 1/2$, we need:

$$\begin{aligned}
\frac{0.7}{0.3} \exp\left(\frac{(1-y)^2 - y^2}{0.98}\right) &\leq 1 \\
\exp\left(\frac{(1-y)^2 - y^2}{0.98}\right) &\leq \frac{3}{7} \\
\frac{(1-y)^2 - y^2}{0.98} &\leq \ln \frac{3}{7} \\
1 - 2y &\leq 0.98 \ln \frac{3}{7}
\end{aligned}$$

which gives the condition

$$y \geq \frac{1}{2}\left(1 - 0.98 \ln \frac{3}{7}\right) \approx 0.915.$$

So, the new rule is to assume that your love interest loves you back if and only if you observe a message which is ≥ 0.915.

(d) As in the first part,

$$\begin{aligned}
&\mathbb{P}(X=0)\mathbb{P}(Y \le 0.915 \mid X=0) + \mathbb{P}(X=1)\mathbb{P}(Y > 0.915 \mid X=1) \\
&\quad = 0.7\mathbb{P}(\mathscr{N}(0,0.49) \le 0.915) + 0.3\mathbb{P}(\mathscr{N}(1,0.49) > 0.915) \\
&\quad = 0.7\mathbb{P}\left(\mathscr{N}(0,1) \le \frac{0.915}{0.7}\right) + 0.3\mathbb{P}\left(\mathscr{N}(0,1) > -\frac{0.085}{0.7}\right) \\
&\quad = 0.7\Phi\left(\frac{0.915}{0.7}\right) + 0.3\Phi\left(\frac{0.085}{0.7}\right) \approx 0.798.
\end{aligned}$$

As you can see, this strategy performs better than the first part.