

1 Counting, Counting, and More Counting

The only way to learn counting is to practice, practice, practice, so here is your chance to do so. Although there are many subparts, each subpart is fairly short, so this problem should not take any longer than a normal CS70 homework problem. You do not need to show work, and **Leave your answers as an expression** (rather than trying to evaluate it to get a specific number).

- (a) How many ways are there to arrange n 1s and k 0s into a sequence?
- (b) How many 7-digit ternary (0,1,2) bitstrings are there such that no two adjacent digits are equal?
- (c) A bridge hand is obtained by selecting 13 cards from a standard 52-card deck. The order of the cards in a bridge hand is irrelevant.
 - i. How many different 13-card bridge hands are there?
 - ii. How many different 13-card bridge hands are there that contain no aces?
 - iii. How many different 13-card bridge hands are there that contain all four aces?
 - iv. How many different 13-card bridge hands are there that contain exactly 6 spades?
- (d) Two identical decks of 52 cards are mixed together, yielding a stack of 104 cards. How many different ways are there to order this stack of 104 cards?
- (e) How many 99-bit strings are there that contain more ones than zeros?
- (f) An anagram of ALABAMA is any re-ordering of the letters of ALABAMA, i.e., any string made up of the letters A, L, A, B, A, M, and A, in any order. The anagram does not have to be an English word.
 - i. How many different anagrams of ALABAMA are there?
 - ii. How many different anagrams of MONTANA are there?
- (g) How many different anagrams of ABCDEF are there if: (1) C is the left neighbor of E; (2) C is on the left of E (and not necessarily E's neighbor)
- (h) We have 9 balls, numbered 1 through 9, and 27 bins. How many different ways are there to distribute these 9 balls among the 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).
- (i) How many different ways are there to throw 9 identical balls into 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).
- (j) We throw 9 identical balls into 7 bins. How many different ways are there to distribute these 9 balls among the 7 bins such that no bin is empty? Assume the bins are distinguishable (e.g., numbered 1 through 7).

- (k) There are exactly 20 students currently enrolled in a class. How many different ways are there to pair up the 20 students, so that each student is paired with one other student? Solve this in at least 2 different ways. **Your final answer must consist of two different expressions.**
- (l) How many solutions does $x_0 + x_1 + \dots + x_k = n$ have, if each x must be a non-negative integer?
- (m) How many solutions does $x_0 + x_1 = n$ have, if each x must be a *strictly positive* integer?
- (n) How many solutions does $x_0 + x_1 + \dots + x_k = n$ have, if each x must be a *strictly positive* integer?

Solution:

- (a) $\binom{n+k}{k}$
- (b) There are 3 options for the first digit. For each of the next digits, they only have 2 options because they cannot be equal to the previous digit. Thus, $3 * 2^6$
- (c) We have to choose 13 cards out of 52 cards, so this is just $\binom{52}{13}$.

We now have to choose 13 cards out of 48 non-ace cards. So this is $\binom{48}{13}$.

We now require the four aces to be present. So we have to choose the remaining 9 cards in our hand from the 48 non-ace cards, and this is $\binom{48}{9}$.

We need our hand to contain 6 out of the 13 spade cards, and 7 out of the 39 non-spade cards, and these choices can be made separately. Hence, there are $\binom{13}{6} \binom{39}{7}$ ways to make up the hand.

- (d) If we consider the $104!$ rearrangements of 2 identical decks, since each card appears twice, we would have overcounted each distinct rearrangement. Consider any distinct rearrangement of the 2 identical decks of 52 cards and see how many times this appears among the rearrangement of 104 cards where each card is treated as different. For each identical pair (such as the two Ace of spades), there are two ways they could be permuted among each other (since $2! = 2$). This holds for each of the 52 pairs of identical cards. So the number $104!$ overcounts the actual number of rearrangements of 2 identical decks by a factor of 2^{52} . Hence, the actual number of rearrangements of 2 identical decks is $104!/2^{52}$.

- (e) **Answer 1:** There are $\binom{99}{k}$ 99-bit strings with k ones and $99 - k$ zeros. We need $k > 99 - k$, i.e. $k \geq 50$. So the total number of such strings is $\sum_{k=50}^{99} \binom{99}{k}$. This expression can however be simplified. Since $\binom{99}{k} = \binom{99}{99-k}$, we have

$$\sum_{k=50}^{99} \binom{99}{k} = \sum_{k=50}^{99} \binom{99}{99-k} = \sum_{l=0}^{49} \binom{99}{l}$$

by substituting $l = 99 - k$. Now $\sum_{k=50}^{99} \binom{99}{k} + \sum_{l=0}^{49} \binom{99}{l} = \sum_{m=0}^{99} \binom{99}{m} = 2^{99}$. Hence, $\sum_{k=50}^{99} \binom{99}{k} = (1/2) \cdot 2^{99} = 2^{98}$.

Answer 2: Symmetry Since the answer from above looked so simple, there must have been a more elegant way to arrive at it. Since 99 is odd, no 99-bit string can have the same number of zeros and ones. Let A be the set of 99-bit strings with more ones than zeros, and B be the set of 99-bit strings with more zeros than ones. Now take any 99-bit string x with more ones than zeros i.e. $x \in A$. If all the bits of x are flipped, then you get a string y with more zeros than ones, and so $y \in B$. This operation of bit flips creates a one-to-one and onto function (called a bijection) between A and B . Hence, it must be that $|A| = |B|$. Every 99-bit string is either in A or in B , and since there are 2^{99} 99-bit strings, we get $|A| = |B| = (1/2) \cdot 2^{99}$. The answer we sought was $|A| = 2^{98}$.

- (f) ALABAMA: The number of ways of rearranging 7 distinct letters and is $7!$. In this 7 letter word, the letter A is repeated 4 times while the other letters appear once. Hence, the number $7!$ overcounts the number of different anagrams by a factor of $4!$ (which is the number of ways of permuting the 4 A's among themselves). Aka, we only want $1/4!$ out of the total rearrangements. Hence, there are $7!/4!$ anagrams.

MONTANA: In this 7 letter word, the letter A and N are each repeated 2 times while the other letters appear once. Hence, the number $7!$ overcounts the number of different anagrams by a factor of $2! \times 2!$ (one factor of $2!$ for the number of ways of permuting the 2 A's among themselves and another factor of $2!$ for the number of ways of permuting the 2 N's among themselves). Hence, there are $7!/(2!)^2$ different anagrams.

- (g) (1) We consider CE is a new letter X, then the question becomes counting the rearranging of 5 distinct letters, and is $5!$. (2) Symmetry: Let A be the set of all the rearranging of ABCDEF with C on the left side of E, and B be the set of all the rearranging of ABCDEF with C on the right side of E. $|A \cup B| = 6!$, $|A \cap B| = 0$. There is a bijection between A and B by construct a operation of exchange the position of C and E. Thus $|A| = |B| = 6!/2$.
- (h) Each ball has a choice of which bin it should go to. So each ball has 27 choices and the 9 balls can make their choices separately. Hence, there are 27^9 ways.
- (i) Since there is no restriction on how many balls a bin needs to have, this is just the problem of throwing k identical balls into n distinguishable bins, which can be done in $\binom{n+k-1}{k}$ ways. Here $k = 9$ and $n = 27$, so there are $\binom{35}{9}$ ways.
- (j) **Answer 1:** Since each bin is required to be non-empty, let's throw one ball into each bin at the outset. Now we have 2 identical balls left which we want to throw into 7 distinguishable bins. There are 2 cases to consider:
Case 1: The 2 balls land in the same bin. This gives 7 ways.
Case 2: The 2 balls land in different bins. This gives $\binom{7}{2}$ ways of choosing 2 out of the 7 bins for the balls to land in. Note that it is *not* 7×6 since the balls are identical and so there is no order on them.
 Summing up the number of ways from both cases, we get $7 + \binom{7}{2}$ ways.

Answer 2: Since each bin is required to be non-empty, let's throw one ball into each bin at the outset. Now we have 2 identical balls left which we want to throw into 7 distinguishable bins.

From class (see note 11), we already saw that the number of ways to put k identical balls into n distinguishable bins is $\binom{n+k-1}{k}$. Taking $k = 2$ and $n = 7$, we get $\binom{8}{2}$ ways to do this.

EASY EXERCISE: Can you give an expression for the number of ways to put k identical balls into n distinguishable bins such that no bin is empty?

- (k) **Answer 1:** Let's number the students from 1 to 20. Student 1 has 19 choices for her partner. Let i be the smallest index among students who have not yet been assigned partners. Then no matter what the value of i is (in particular, i could be 2 or 3), student i has 17 choices for her partner. The next smallest indexed student who doesn't have a partner now has 15 choices for her partner. Continuing in this way, the number of pairings is $19 \times 17 \times 15 \times \dots \times 1 = \prod_{i=1}^{10} (2i - 1)$.

Answer 2: Arrange the students numbered 1 to 20 in a line. There are $20!$ such arrangements. We pair up the students at positions $2i - 1$ and $2i$ for i ranging from 1 to 10. You should be able to see that the $20!$ permutations of the students doesn't miss any possible pairing. However, it counts every different pairing multiple times. Fix any particular pairing of students. In this pairing, the first pair had freedom of 10 positions in any permutation that generated it, the second pair had a freedom of 9 positions in any permutation that generated it, and so on. There is also the freedom for the elements within each pair i.e. in any student pair (x, y) , student x could have appeared in position $2i - 1$ and student y could have appeared in position $2i$ and also vice versa. This gives 2 ways for each of the 10 pairs. Thus, in total, these freedoms cause $10! \times 2^{10}$ of the $20!$ permutations to give rise to this particular pairing. This holds for each of the different pairings. Hence, $20!$ overcounts the number of different pairings by a factor of $10! \times 2^{10}$. Hence, there are $20! / (10! \cdot 2^{10})$ pairings.

Answer 3: In the first step, pick a pair of students from the 20 students. There are $\binom{20}{2}$ ways to do this. In the second step, pick a pair of students from the remaining 18 students. There are $\binom{18}{2}$ ways to do this. Keep picking pairs like this, until in the tenth step, you pick a pair of students from the remaining 2 students. There are $\binom{2}{2}$ ways to do this. Multiplying all these, we get $\binom{20}{2} \binom{18}{2} \dots \binom{2}{2}$. However, in any particular pairing of 20 students, this pairing could have been generated in $10!$ ways using the above procedure depending on which pairs in the pairing got picked in the first step, second step, \dots , tenth step. Hence, we have to divide the above number by $10!$ to get the number of different pairings. Thus there are $\binom{20}{2} \binom{18}{2} \dots \binom{2}{2} / 10!$ different pairings of 20 students.

You may want to check for yourself that all three methods are producing the same integer, even though they are expressed very differently.

- (l) $\binom{n+k}{k}$. This is just n indistinguishable balls into $k+1$ distinguishable bins (stars and bars). There is a bijection between a sequence of n ones and k plusses and a solution to the equation: x_0 is the number of ones before the first plus, x_1 is the number of ones between the first and second plus, etc. A key idea is that if a bijection exists between two sets they must be the same size, so counting the elements of one tells us how many the other has. Note that this is the exact same answer as part a - make sure you understand why!

- (m) $n - 1$. It's easy just to enumerate the solutions here. x_0 can take values $1, 2, \dots, n - 1$ and this

uniquely fixes the value of x_1 . So, we have $n - 1$ ways to do this. But, this is just an example of the more general question below.

- (n) $\binom{(n-(k+1))+k}{k} = \binom{n-1}{k}$. This is just $n-(k+1)$ indistinguishable balls into distinguishable $k+1$ bins. By subtracting 1 from all $k+1$ variables, and $k+1$ from the total required, we reduce it to problem with the same form as the previous problem. Once we have a solution to that we reverse the process, and adding 1 to all the non-negative variables gives us positive variables.

2 Counting on Graphs + Symmetry

- (a) How many distinct undirected graphs are there with n labeled vertices? Assume that there can be at most one edge between any two vertices, and there are no edges from a vertex to itself. The graphs do not have to be connected.
- (b) How many distinct cycles are there in a complete graph K_n with n vertices? Assume that cycles cannot have duplicated edges. Two cycles are considered the same if they are rotations or inversions of each other (e.g. (v_1, v_2, v_3, v_1) , (v_2, v_3, v_1, v_2) and (v_1, v_3, v_2, v_1) all count as the same cycle).
- (c) How many ways are there to color a bracelet with n beads using n colors, such that each bead has a different color? Note: two colorings are considered the same if one of them can be obtained by rotating the other.
- (d) How many ways are there to color the faces of a cube using exactly 6 colors, such that each face has a different color? Note: two colorings are considered the same if one can be obtained from the other by rotating the cube in any way.

Solution:

- (a) There are $\binom{n}{2} = n(n-1)/2$ possible edges, and each edge is either present or not. So the answer is $2^{n(n-1)/2}$. (Recall that $2^m = \sum_{k=0}^m \binom{m}{k}$, where $m = n(n-1)/2$ in this case.)
- (b) The number k of vertices in a cycle is at least 3 and at most n . Without accounting for duplicates, there are $n!/(n-k)!$ cycles. Due to inversions (read from front or end doesn't matter, $abc = cba$) we divide by 2, and rotations (where we start reading from doesn't matter $abc = bca = cab$) we divide by k . Hence the total number of distinct cycles is

$$\sum_{k=3}^n \frac{n!}{(n-k)! \cdot 2k}.$$

- (c) Without considering symmetries there are $n!$ ways to color the beads on the bracelet. Due to rotations, there are n equivalent colorings for any given coloring. Hence taking into account symmetries, there are $(n-1)!$ distinct colorings. Note: if in addition to rotations, we also consider flips/mirror images, then the answer would be $(n-1)!/2$.

- (d) Without considering symmetries there are $6!$ ways to color the faces of the cube. The number of equivalent colorings, for any given coloring, is $24 = 6 \times 4$: 6 comes from the fact that every given face can be rotated to face any of the six directions. 4 comes from the fact that after we decide the direction of a certain face, we can rotate the cube around this axis in 4 different ways (including no further rotations). Hence there are $6!/24 = 30$ distinct colorings.

3 That's Numberwang!

Congratulations! You've earned a spot on the game show "Numberwang".

- (a) How many permutations of NUMBERWANG contain "GAME" as a substring? How about as a subsequence (meaning the letters of "GAME" have to appear in that order, but not necessarily next to each other)?
- (b) In round 1 of Numberwang, each player chooses an ordered sequence of 5 digits. A valid sequence must have the property that it is non-increasing when read from left to right. For example, 99621 is a valid sequence, but 43212 is not. How many choices of valid sequences are there? (**Hint:** Relate the problem to balls and bins.)
- (c) To win round 2 of Numberwang, a contestant must choose five nonnegative integers x_0, x_1, x_2, x_3, x_4 such that $x_0 + x_1 + x_2 + x_3 + x_4 = 100$, and $x_i \equiv i \pmod{5}$. How many ways are there to pick a winning set of integers?

Solution:

- (a) If we need "GAME" to be a substring, then this is equivalent to finding the number of ways to arrange 7 items in a row ("GAME", "N", "U", "B", "R", "W", and "N"), where two of the items are the same (the two "N"s). This amounts to $7!/2! = 2520$ ways.

If we need "GAME" to be a subsequence, let's first choose which 4 positions contain the letters of GAME; then, that uniquely determines the positions of G, A, M, E, since they have to be in that order. There are $\binom{10}{4}$ ways to do this, and $6!/2!$ ways to choose the positions of the remaining six letters (two of which are the same), so there are a total of $\binom{10}{4} \frac{6!}{2!} = 75600$ valid subsequences.

- (b) This is actually a "balls and bins" (or "stars and bars") problem in disguise! We have five digits ("balls"), and 10 "bins" describing the values of the digits from 0 to 9: one bin for nines, one bin for eights, etc. This is because we know that the digits are non-increasing, so all the nines (if any) must come first, then all the eights (if any), and so on. So we just need to count how many ways there are to distribute the digits into the 10 bins: so our answer is $\binom{10+5-1}{10-1} = \binom{14}{9}$.
- (c) Let $x_i = 5y_i + i$ for nonnegative integers y_i (we do this because of the modulo conditions). Then, the equation we must satisfy becomes $(5y_0 + 0) + (5y_1 + 1) + (5y_2 + 2) + (5y_3 + 3) + (5y_4 + 4) = 100$, which simplifies down to $y_0 + y_1 + y_2 + y_3 + y_4 = 18$ for nonnegative integers

y_i . This is a standard balls and bins (or stars and bars) problem, with 18 balls and 5 bins, so our answer is $\binom{22}{4}$.

Why is $y_0 + y_1 + y_2 + y_3 + y_4 = 18$ a balls and bins problem? y_i is the number of balls in bin i , and 18 is the number of balls in total. Here are a few possible arrangements, with *s representing balls:

$$\underbrace{*****}_6 | \underbrace{\quad}_0 | \underbrace{*****}_7 | \underbrace{*****}_5 | \underbrace{\quad}_0$$

$$\underbrace{**}_2 | \underbrace{****}_4 | \underbrace{****}_4 | \underbrace{*****}_6 | \underbrace{**}_2$$

Notice that we arrange the balls into 5 bins using 4 dividers. The total number of ways we can arrange 18 balls and 4 dividers side-by-side in a line is $\binom{22}{4}$. Why?

- 22 is the total number of objects in the line, 18 balls + 4 dividers.
- 4 is the number of dividers to be placed in the line.

“22 choose 4”, or $\binom{22}{4}$, is the number of ways we can position 4 dividers on a line with 18 indistinguishable balls.

4 Captain Combinatorial

Please provide combinatorial proofs for the following identities.

- (a) $\binom{n}{i} = \binom{n}{n-i}$.
- (b) $\sum_{i=1}^n i \binom{n}{i}^2 = n \binom{2n-1}{n-1}$. (Hint: Part (a) might be useful.)
- (c) $\sum_{i=0}^n \binom{n}{i} \sum_{j=0}^{n-i} \binom{n-i}{j} = 3^n$. (Hint: consider the number of ways of splitting n elements into 3 groups.)

Solution:

- (a) Choosing i players out of n to play on a team is the same as choosing $n - i$ players to not play on the team, i.e. $\binom{n}{i} = \binom{n}{n-i}$.
- (b) Assume we have n women and n men. Using part (a) we can rewrite the LHS as $\sum_{i=1}^n i \binom{n}{i} \binom{n}{n-i}$, which we can interpret as selecting a team of n players by choosing i women and $n - i$ men, and then choosing one of the women to serve as captain. Again, the RHS first chooses a captain, and then selects a remaining $n - 1$ players from all remaining men and women to form the team.

- (c) We count the number of ways to split n elements into 3 labeled groups by two different methods.

RHS: There are 3 different choices for each element, so 3^n for all of them.

LHS: For every i from 0 to n , choose i elements to go in group A, then for every j from 0 to $n - i$ choose j elements to go in group B, the remaining go in group C. This gives:

$$\sum_{i=0}^n \binom{n}{i} \sum_{j=0}^{n-i} \binom{n-i}{j}$$

5 Fermat's Wristband

Let p be a prime number and let k be a positive integer. We have beads of k different colors, where any two beads of the same color are indistinguishable.

- (a) We place p beads onto a string. How many different ways are there construct such a sequence of p beads with up to k different colors?
- (b) How many sequences of p beads on the string are there that use at least two colors?
- (c) Now we tie the two ends of the string together, forming a wristband. Two wristbands are equivalent if the sequence of colors on one can be obtained by rotating the beads on the other. (For instance, if we have $k = 3$ colors, red (R), green (G), and blue (B), then the length $p = 5$ necklaces RGGBG, GGBGR, GBGRG, BGRGG, and GRGGB are all equivalent, because these are all rotated versions of each other.)

How many non-equivalent wristbands are there now? Again, the p beads must not all have the same color. (Your answer should be a simple function of k and p .)

[*Hint:* Think about the fact that rotating all the beads on the wristband to another position produces an identical wristband.]

- (d) Use your answer to part (c) to prove Fermat's little theorem.

Solution:

- (a) k^p . For each of the p beads, there are k possibilities for its colors. Therefore, by the first counting principle, there are k^p different sequences.
- (b) $k^p - k$. You can have k sequences of a beads with only one color.
- (c) Since p is prime, rotating any sequence by less than p spots will produce a new sequence. As in, there is no number x smaller than p such that rotating the beads by x would cause the pattern to look the same. So, every pattern which has more than one color of beads can be rotated to form $p - 1$ other patterns. So the total number of patterns equivalent with some bead sequence is p . Thus, the total number of non-equivalent patterns are $(k^p - k)/p$.

(d) $(k^p - k)/p$ must be an integer, because from the previous part, it is the number of ways to count something. Hence, $k^p - k$ has to be divisible by p , i.e., $k^p \equiv k \pmod{p}$, which is Fermat's Little Theorem.