

Due: Friday, 03/05 at 10:00 PM
Grace period until Friday, 03/05 at 11:59 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Counting, Counting, and More Counting

The only way to learn counting is to practice, practice, practice, so here is your chance to do so. Although there are many subparts, each subpart is fairly short, so this problem should not take any longer than a normal CS70 homework problem. You do not need to show work, and **Leave your answers as an expression** (rather than trying to evaluate it to get a specific number).

- (a) How many ways are there to arrange n 1s and k 0s into a sequence?
- (b) How many 7-digit ternary (0,1,2) bitstrings are there such that no two adjacent digits are equal?
- (c) A bridge hand is obtained by selecting 13 cards from a standard 52-card deck. The order of the cards in a bridge hand is irrelevant.
 - i. How many different 13-card bridge hands are there?
 - ii. How many different 13-card bridge hands are there that contain no aces?
 - iii. How many different 13-card bridge hands are there that contain all four aces?
 - iv. How many different 13-card bridge hands are there that contain exactly 6 spades?
- (d) Two identical decks of 52 cards are mixed together, yielding a stack of 104 cards. How many different ways are there to order this stack of 104 cards?
- (e) How many 99-bit strings are there that contain more ones than zeros?
- (f) An anagram of ALABAMA is any re-ordering of the letters of ALABAMA, i.e., any string made up of the letters A, L, A, B, A, M, and A, in any order. The anagram does not have to be an English word.
 - i. How many different anagrams of ALABAMA are there?
 - ii. How many different anagrams of MONTANA are there?

- (g) How many different anagrams of ABCDEF are there if: (1) C is the left neighbor of E; (2) C is on the left of E (and not necessarily E's neighbor)
- (h) We have 9 balls, numbered 1 through 9, and 27 bins. How many different ways are there to distribute these 9 balls among the 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).
- (i) How many different ways are there to throw 9 identical balls into 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).
- (j) We throw 9 identical balls into 7 bins. How many different ways are there to distribute these 9 balls among the 7 bins such that no bin is empty? Assume the bins are distinguishable (e.g., numbered 1 through 7).
- (k) There are exactly 20 students currently enrolled in a class. How many different ways are there to pair up the 20 students, so that each student is paired with one other student? Solve this in at least 2 different ways. **Your final answer must consist of two different expressions.**
- (l) How many solutions does $x_0 + x_1 + \cdots + x_k = n$ have, if each x must be a non-negative integer?
- (m) How many solutions does $x_0 + x_1 = n$ have, if each x must be a *strictly positive* integer?
- (n) How many solutions does $x_0 + x_1 + \cdots + x_k = n$ have, if each x must be a *strictly positive* integer?

2 Counting on Graphs + Symmetry

- (a) How many distinct undirected graphs are there with n labeled vertices? Assume that there can be at most one edge between any two vertices, and there are no edges from a vertex to itself. The graphs do not have to be connected.
- (b) How many distinct cycles are there in a complete graph K_n with n vertices? Assume that cycles cannot have duplicated edges. Two cycles are considered the same if they are rotations or inversions of each other (e.g. (v_1, v_2, v_3, v_1) , (v_2, v_3, v_1, v_2) and (v_1, v_3, v_2, v_1) all count as the same cycle).
- (c) How many ways are there to color a bracelet with n beads using n colors, such that each bead has a different color? Note: two colorings are considered the same if one of them can be obtained by rotating the other.
- (d) How many ways are there to color the faces of a cube using exactly 6 colors, such that each face has a different color? Note: two colorings are considered the same if one can be obtained from the other by rotating the cube in any way.

3 That's Numberwang!

Congratulations! You've earned a spot on the game show "Numberwang".

- (a) How many permutations of NUMBERWANG contain "GAME" as a substring? How about as a subsequence (meaning the letters of "GAME" have to appear in that order, but not necessarily next to each other)?
- (b) In round 1 of Numberwang, each player chooses an ordered sequence of 5 digits. A valid sequence must have the property that it is non-increasing when read from left to right. For example, 99621 is a valid sequence, but 43212 is not. How many choices of valid sequences are there? (**Hint:** Relate the problem to balls and bins.)
- (c) To win round 2 of Numberwang, a contestant must choose five nonnegative integers x_0, x_1, x_2, x_3, x_4 such that $x_0 + x_1 + x_2 + x_3 + x_4 = 100$, and $x_i \equiv i \pmod{5}$. How many ways are there to pick a winning set of integers?

4 Captain Combinatorial

Please provide combinatorial proofs for the following identities.

- (a) $\binom{n}{i} = \binom{n}{n-i}$.
- (b) $\sum_{i=1}^n i \binom{n}{i}^2 = n \binom{2n-1}{n-1}$. (Hint: Part (a) might be useful.)
- (c) $\sum_{i=0}^n \binom{n}{i} \sum_{j=0}^{n-i} \binom{n-i}{j} = 3^n$. (Hint: consider the number of ways of splitting n elements into 3 groups.)

5 Fermat's Wristband

Let p be a prime number and let k be a positive integer. We have beads of k different colors, where any two beads of the same color are indistinguishable.

- (a) We place p beads onto a string. How many different ways are there to construct such a sequence of p beads with up to k different colors?
- (b) How many sequences of p beads on the string are there that use at least two colors?
- (c) Now we tie the two ends of the string together, forming a wristband. Two wristbands are equivalent if the sequence of colors on one can be obtained by rotating the beads on the other. (For instance, if we have $k = 3$ colors, red (R), green (G), and blue (B), then the length $p = 5$ necklaces RGGGB, GGBGR, GBGRG, BGRGG, and GRGGB are all equivalent, because these are all rotated versions of each other.)

How many non-equivalent wristbands are there now? Again, the p beads must not all have the same color. (Your answer should be a simple function of k and p .)

[Hint: Think about the fact that rotating all the beads on the wristband to another position produces an identical wristband.]

- (d) Use your answer to part (c) to prove Fermat's little theorem.