

Due: Friday, 03/19 at 10:00 PM
Grace period until Sunday, 03/28 at 11:59 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Independent Complements

Let Ω be a sample space, and let $A, B \subseteq \Omega$ be two independent events.

- (a) Prove or disprove: \bar{A} and \bar{B} must be independent.
- (b) Prove or disprove: A and \bar{B} must be independent.
- (c) Prove or disprove: A and \bar{A} must be independent.
- (d) Prove or disprove: It is possible that $A = B$.

2 Probability Warm-Up

- (a) Suppose that we have a bucket of 30 red balls and 70 blue balls. If we pick 20 balls out of the bucket, what is the probability of getting exactly k red balls (assuming $0 \leq k \leq 20$) if the sampling is done **with** replacement, i.e. after we take a ball out the bucket we return the ball back to the bucket for the next round?
- (b) Same as part (a), but the sampling is **without** replacement, i.e. after we take a ball out the bucket we **do not** return the ball back to the bucket.
- (c) If we roll a regular, 6-sided die 5 times. What is the probability that at least one value is observed more than once?

3 Past Probabilified

In this question we review some of the past CS70 topics, and look at them probabilistically. For the following experiments,

- i. Define an appropriate sample space Ω .
 - ii. Give the probability function $\mathbb{P}(\omega)$.
 - iii. Compute $\mathbb{P}(E_1)$ given event E_1 .
 - iv. Compute $\mathbb{P}(E_2)$ given event E_2 .
- (a) Fix a prime $p > 2$, and uniformly sample twice with replacement from $\{0, \dots, p-1\}$ (assume we have two $\{0, \dots, p-1\}$ -sided fair dice and we roll them). Then multiply these two numbers with each other in $(\text{mod } p)$ space.
 $E_1 =$ The resulting product is 0.
 $E_2 =$ The product is $(p-1)/2$.
 - (b) Make a graph on n vertices by sampling uniformly at random from all possible edges, (assume for each edge we flip a coin and if it is head we include the edge in the graph and otherwise we exclude that edge from the graph).
 $E_1 =$ The graph is complete.
 $E_2 =$ vertex v_1 has degree d .

4 Peaceful rooks

A friend of yours, Eithen Quinn, is fascinated by the following problem: placing m rooks on an $n \times n$ chessboard, so that they are in peaceful harmony (i.e. no two threaten each other). Each rook is a chess piece, and two rooks threaten each other if and only if they are in the same row or column. You remind your friend that this is so simple that a baby can accomplish the task. You forget however that babies cannot understand instructions, so when you give the m rooks to your baby niece, she simply puts them on random places on the chessboard. She however, never puts two rooks at the same place on the board.

- (a) Assuming your niece picks the places uniformly at random, what is the chance that she places the $(i+1)^{\text{st}}$ rook such that it doesn't threaten any of the first i rooks, given that the first i rooks don't threaten each other?
- (b) What is the chance that your niece actually accomplishes the task and does not prove you wrong?
- (c) Now imagine that the rooks can be stacked on top of each other, then what would be the probability that your niece's placements result in peace? Assume that two rooks threaten each other if they are in the same row or column. Also two pieces stacked on top of each other are obviously in the same row and column, therefore they threaten each other.
- (d) Explain the relationship between your answer to the previous part and the birthday paradox. In particular if we assume that 23 people have a 50% chance of having a repeated birthday (in a 365-day calendar), what is the probability that your niece places 23 stackable pieces in a peaceful position on a 365×365 board?

5 Five Up

Say you toss a coin five times, and record the outcomes. For the three questions below, you can assume that order matters in the outcome, and that the probability of heads is some p in $0 < p < 1$, but *not* that the coin is fair ($p = 0.5$).

- (a) What is the size of the sample space, $|\Omega|$?
- (b) How many elements of Ω have exactly three heads?
- (c) How many elements of Ω have three or more heads?
(*Hint: Argue by symmetry.*)

For the next three questions, you can assume that the coin is fair (i.e. heads comes up with $p = 0.5$, and tails otherwise).

- (d) What is the probability that you will observe the sequence HHHTT? What about HHHHT?
- (e) What is the chance of observing at least one head?
- (f) What about the chance of observing three or more heads?

For the final three questions, you can instead assume the coin is biased so that it comes up heads with probability $p = \frac{2}{3}$.

- (g) What is the chance of observing the outcome HHHTT? What about HHHHT?
- (h) What about the chance of at least one head?
- (i) What about the chance of ≥ 3 heads?

6 Ball-and-Bin Counting Problems

Say you have 5 bins, and randomly throw 7 balls into them.

1. What is the probability that the first bin has precisely 3 balls in it?
2. What is the probability that the third bin has at least 3 balls in it?
3. What is the probability that at least one of the bins has precisely 3 balls in it?