Today.

Last time:
Today.

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  Shared (and sort of kept) secrets.
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Last time:
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Today: Errors
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Today: Errors  
  Tolerate Loss: erasure codes.
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  Tolerate corruption!
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Today: Errors
  Tolerate Loss: erasure codes.
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The mathematics.

There is a unique polynomial of degree $d$ that contains any $d + 1$ points.
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Assumption: a field, in particular, arithmetic $\mod p$. 

Big Idea: 

A polynomial: 

$$P(x) = a_d x^d + \cdots + a_0$$ 

has $d + 1$ coefficients.

Any set of $d + 1$ points determines the polynomial.
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Some details:
Degree $d$ generally degree “at most” $d$.
(example: choose 10 points on a line.)
Arithmetic $\mod p \implies$ work with $O(\log p)$ bit numbers.
In general.

Given points: \((x_1, y_1); (x_2, y_2) \cdots (x_k, y_k)\).
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\[
\Delta_i(x) = \frac{\prod_{j \neq i} (x - x_j)}{\prod_{j \neq i} (x_i - x_j)}
\]
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The numerator is 0 at $x_j \neq x_i$.

The denominator makes it 1 at $x_i$.

And...

$$P(x) = y_1 \Delta_1(x) + y_2 \Delta_2(x) + \cdots + y_k \Delta_k(x).$$

This construction proves the existence of the polynomial!
In general.

Given points: \((x_1, y_1); (x_2, y_2) \cdots (x_k, y_k)\).

\[
\Delta_i(x) = \frac{\prod_{j \neq i} (x - x_j)}{\prod_{j \neq i} (x_i - x_j)} = \frac{\prod (x - x_j)}{\prod (x_i - x_j)} \prod (x_i - x_j)^{-1}
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Construction proves the existence of the polynomial!
Uniqueness.

**Uniqueness Fact.** At most one degree $d$ polynomial hits $d + 1$ points.
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**Proof:**
Assume two different polynomials $Q(x)$ and $P(x)$ hit the points.
Uniqueness Fact. At most one degree $d$ polynomial hits $d + 1$ points.

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Non-zero line (degree 1 polynomial) can intersect $y = 0$ at only one $x$.
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Proof:
Assume two different polynomials $Q(x)$ and $P(x)$ hit the points.
$R(x) = Q(x) - P(x)$ has $d + 1$ roots and is degree $d$. 
**Uniqueness Fact.** At most one degree $d$ polynomial hits $d + 1$ points.

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Assume two different polynomials $Q(x)$ and $P(x)$ hit the points.

$$R(x) = Q(x) - P(x)$$ has $d + 1$ roots and is degree $d$. Contradiction.
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Contradiction.

Must prove Roots fact.
Polynomial Division.

Divide $4x^2 - 3x + 2$ by $(x - 3)$ modulo 5.
Polynomial Division.

Divide $4x^2 - 3x + 2$ by $(x - 3)$ modulo 5.

$$
\begin{array}{c}
4 \\
\hline
x - 3 \quad 4x^2 - 3x + 2
\end{array}
$$

$$
\begin{array}{c}
\quad \\
\hline
\quad \\
4x^2 - 2x
\end{array}
$$

$$
\begin{array}{c}
\quad \\
\hline
\quad \\
4x + 2
\end{array}
$$

$$
\begin{array}{c}
\quad \\
\hline
\quad \\
4x - 2
\end{array}
$$

$$
\begin{array}{c}
\quad \\
\hline
\quad \\
4
\end{array}
$$

$$
4x^2 - 3x + 2 \equiv (x - 3)(4x + 4) + 4 \pmod{5}
$$

In general, divide $P(x)$ by $(x - a)$ gives $Q(x)$ and remainder $r$.

That is, $P(x) = (x - a)Q(x) + r$. 

Polynomial Division.

Divide $4x^2 - 3x + 2$ by $(x - 3)$ modulo 5.

\[
\begin{array}{r}
4 & x \\
\hline
x - 3 & 4x^2 - 3x + 2 \\
& 4x^2 - 2x \\
& \underline{4x} \\
& 4 \\
\end{array}
\]

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Divide $4x^2 - 3x + 2$ by $(x - 3)$ modulo 5.

\[
\begin{array}{c|cc}
  & 4x & + 4 \\
\hline
x - 3) & 4x^2 & - 3x + 2 \\
& 4x^2 & - 2x \\
\hline
& 4x & + 2
\end{array}
\]

In general, divide $P(x)$ by $(x - a)$ gives $Q(x)$ and remainder $r$.

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Polynomial Division.

Divide $4x^2 - 3x + 2$ by $(x - 3)$ modulo 5.

\[
\begin{array}{c|cc}
  & 4x + 4 \\
\hline
x - 3 & 4x^2 - 3x + 2 \\
      & 4x^2 - 2x \\
      & 4x + 2 \\
\end{array}
\]

\[
4x + 2
\]

\[
4x - 2
\]

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\[
\begin{array}{c|cc}
\multicolumn{3}{c}{4x + 4} \\
\hline
x - 3 & 4x^2 - 3x + 2 \\
\hline
& 4x^2 - 2x \\
\hline
& 4x + 2 \\
\hline
& 4x - 2 \\
\hline
& 4
\end{array}
\]

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$$
\begin{array}{cccc}
\phantom{+4} & 4 & x & + 4 & r & 4 \\
\hline
4x - 3 & \smash{\overline{4x^2 - 3x + 2}} \\
\hline
\phantom{+4} & 4x^2 - 2x \\
\hline
\phantom{+4} & 4x + 2 \\
\phantom{+4} & 4x - 2 \\
\hline
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\hline
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\hline
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   & 4x - 2 \\
\hline
   & 4 \\
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$$

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$$\begin{array}{c|cccc}
4 & x & + & 4 & r & 4 \\
\hline
x & - & 3 & ) & 4x^2 & - & 3x & + & 2 \\
4x^2 & - & 2x & \\
\hline
4x & + & 2 & \\
4x & - & 2 & \\
\hline
4 & \\
\end{array}$$

$$4x^2 - 3x + 2 \equiv (x - 3)(4x + 4) + 4 \pmod{5}$$

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\hline 
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\hline 
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\hline 
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In general, divide $P(x)$ by $(x - a)$ gives $Q(x)$ and remainder $r$.

That is, $P(x) = (x - a)Q(x) + r$
Only $d$ roots.

**Lemma 1:** $P(x)$ has root $a$ iff $P(x)/(x - a)$ has remainder 0: $P(x) = (x - a)Q(x)$. 

Proof: 

$$P(x) = (x - a)Q(x) + r.$$ 

Plugin $a$: $P(a) = r$. It is a root if and only if $r = 0$.
Lemma 1: $P(x)$ has root $a$ iff $P(x)/(x - a)$ has remainder 0: $P(x) = (x - a)Q(x)$.

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Lemma 2: \( P(x) \) has \( d \) roots; \( r_1, \ldots, r_d \) then
\[
P(x) = c(x)(x - r_1)(x - r_2) \cdots (x - r_d).
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**Proof Sketch:** By induction.
Lemma 1: $P(x)$ has root $a$ iff $P(x)/(x - a)$ has remainder 0: 
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Proof Sketch: By induction.

Induction Step: $P(x) = (x - r_1)Q(x)$ by Lemma 1. $Q(x)$ has smaller degree so use the induction hypothesis.
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Implication: $d + 1$ roots $\rightarrow \geq d + 1$ terms $\Longrightarrow$ degree is $\geq d + 1$. 

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**Roots fact:** Any degree $\leq d$ polynomial has at most $d$ roots.
Finite Fields

Proof works for reals, rationals, and complex numbers.
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Good for computer science.
Finite Fields

Proof works for reals, rationals, and complex numbers. ..but not for integers, since no multiplicative inverses. Arithmetic modulo a prime $p$ has multiplicative inverses.. ..and has only a finite number of elements. Good for computer science. Arithmetic modulo a prime $m$ is a **finite field** denoted by $F_m$ or $GF(m)$. 
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Arithmetic modulo a prime $m$ is a **finite field** denoted by $F_m$ or $GF(m)$.

Intuitively, a field is a set with operations corresponding to addition, multiplication, and division.
Secret Sharing

**Modular Arithmetic Fact:** Exactly one polynomial degree $\leq d$ over $GF(p)$, $P(x)$, that hits $d + 1$ points.
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**Shamir’s \( k \) out of \( n \) Scheme:**

1. Choose \( a_0 = s \), and randomly \( a_1, \ldots, a_{k-1} \).
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3. Share \( i \) is point \((i, P(i) \mod p)\).

**Robustness:** Any \( k \) knows secret. Knowing \( k \) pts, only one \( P(x) \), evaluate \( P(0) \).

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Need $p > n$ to hand out $n$ shares: $P(1) \ldots P(n)$. 
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(Almost) the same as what is missing: one $P(i)$. 
Runtime: polynomial in $k$, $n$, and $\log p$.

1. Evaluate degree $k-1$ polynomial $n$ times using $\log p$-bit numbers.
2. Reconstruct secret by solving system of $k$ equations using $\log p$-bit arithmetic.
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A bit more counting.

What is the number of degree $d$ polynomials over $GF(m)$?
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- $m^{d+1}$: $d + 1$ coefficients from $\{0, \ldots, m - 1\}$. 

Infinite number for reals, rationals, complex numbers!
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Erasure Codes.

Satellite

GPS device
Erasure Codes.

Satellite

3 packet message.

GPS device
Erasure Codes.

Satellite

3 packet message.

GPS device

Lose 3 out 6 packets.
Erasure Codes.

3 packet message. So send 6!

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Satellite

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3 packet message. So send 6!

Lose 3 out 6 packets.

Gets packets 1, 1, and 3.
Solution Idea.

$n$ packet message, channel that loses $k$ packets.
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\( n \) packet message, channel that loses \( k \) packets.
Must send \( n + k \) packets!
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Any $n$ packets
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Use polynomials.
The Scheme

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Erasure Codes.

Satellite

GPS device
Erasure Codes.

Satellite

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Erasure Codes.

Satellite

\[ n \text{ packet message.} \]

Lose \( k \) packets.

GPS device

Optimal.
Erasure Codes.

Satellite

$n$ packet message. So send $n + k$!

Lose $k$ packets.

GPS device
Erasure Codes.

 Satellite

 1 2 ... \( n+k \) \[\begin{array}{c}
\hline
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 GPS device

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Satellite

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Information Theory.

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Send message of 1, 4, and 4.
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Make polynomial with $P(1) = 1$, $P(2) = 4$, $P(3) = 4$. 

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Erasure Code: Example.

Send message of 1, 4, and 4.

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How?

Better work modulo 7 at least!

Why?

$(0, P(0)) = (5, P(5)) \pmod{5}$
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How?

Lagrange Interpolation.
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Linear System.

Work modulo 5.
Send message of 1, 4, and 4.

Make polynomial with $P(1) = 1$, $P(2) = 4$, $P(3) = 4$.

How?

- Lagrange Interpolation.
- Linear System.

Work modulo 5.

$P(x) = x^2 \pmod{5}$
Erasure Code: Example.

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$P(1) = 1$, $P(2) = 4$, $P(3) = 4$.
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Lagrange Interpolation.
Linear System.

Work modulo 5.

$P(x) = x^2 \pmod{5}$

$P(1) = 1$, $P(2) = 4$, $P(3) = 9 = 4 \pmod{5}$
Erasure Code: Example.

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Make polynomial with $P(1) = 1$, $P(2) = 4$, $P(3) = 4$.

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Lagrange Interpolation.
Linear System.

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6 points.
Erasure Code: Example.

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How?

Lagrange Interpolation.
Linear System.

Work modulo 5.

\[ P(x) = x^2 \pmod{5} \]

\[ P(1) = 1, P(2) = 4, P(3) = 9 = 4 \pmod{5} \]

Send \((0, P(0))\ldots(5, P(5))\).

6 points. Better work modulo 7 at least!
Erasure Code: Example.

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Send $(0, P(0)) \ldots (5, P(5))$.

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Why? $(0, P(0)) = (5, P(5)) \pmod{5}$
Example

Make polynomial with $P(1) = 1$, $P(2) = 4$, $P(3) = 4$. 

Send Packets: $(1, 1), (2, 4), (3, 4), (4, 7), (5, 2), (6, 0)$.
Example

Make polynomial with \( P(1) = 1, \ P(2) = 4, \ P(3) = 4. \)
Modulo 7 to accommodate at least 6 packets.
Example

Make polynomial with $P(1) = 1$, $P(2) = 4$, $P(3) = 4$.
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Linear equations:

\begin{align*}
P(1) &= a_2 + a_1 + a_0 \equiv 1 \pmod{7} \\
6a_1 + 3a_0 &= 2 \pmod{7} \\
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\end{align*}

$a_1 = 2 \pmod{7}$

$a_0 = 2 \pmod{7}$

$a_2 = 2 \pmod{7}$

Send packets: $(1, 1)$, $(2, 4)$, $(3, 4)$, $(4, 7)$, $(5, 2)$, $(6, 0)$

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$$a_1 = 2a_0.$$
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$a_1 = 2a_0$. $a_0 = 2 \pmod{7}$ $a_1 = 4 \pmod{7}$ $a_2 = 2 \pmod{7}$

$P(x) = 2x^2 + 4x + 2$
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Send
Example

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Packets: \((1, 1), (2, 4), (3, 4), (4, 7), (5, 2), (6, 0)\)
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\[P(1) = 1, \ P(2) = 4, \ \text{and} \ P(3) = 4\]

Send

Packets: (1, 1), (2, 4), (3, 4), (4, 7), (5, 2), (6, 0)

Notice that packets contain “x-values”.
Bad reception!

Send: (1,1), (2,4), (3,4), (4,7), (5,2), (6,0)
Bad reception!

Send: (1,1), (2,4), (3,4), (4,7), (5,2), (6,0)
Recieve: (1,1) (2,4), (6,0)
Bad reception!

Send: (1,1), (2,4), (3,4), (4,7), (5,2), (6,0)
Recieve: (1,1) (2,4), (6,0)
Reconstruct?
Bad reception!

Send: (1, 1), (2, 4), (3, 4), (4, 7), (5, 2), (6, 0)

Receive: (1, 1), (2, 4), (6, 0)

Reconstruct?

Format: \((i, R(i))\).
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Lagrange or linear equations.
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Channeling Sahai
Bad reception!

Send: \((1, 1), (2, 4), (3, 4), (4, 7), (5, 2), (6, 0)\)

Recieve: \((1, 1)\) \((2, 4)\), \((6, 0)\)

Reconstruct?

Format: \((i, R(i))\).

Lagrange or linear equations.

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\begin{align*}
P(1) &= a_2 + a_1 + a_0 \equiv 1 \pmod{7} \\
P(2) &= 4a_2 + 2a_1 + a_0 \equiv 4 \pmod{7} \\
P(6) &= 2a_2 + 3a_1 + a_0 \equiv 0 \pmod{7}
\end{align*}
\]

Channeling Sahai ...

\[
P(x) = 2x^2 + 4x + 2
\]

Message? \(P(1) = 1\),
Bad reception!

Send: $(1, 1), (2, 4), (3, 4), (4, 7), (5, 2), (6, 0)$

Recieve: $(1,1) (2,4), (6,0)$
  Reconstruct?

Format: $(i, R(i))$.

Lagrange or linear equations.

\[
\begin{align*}
  P(1) &= a_2 + a_1 + a_0 \equiv 1 \pmod{7} \\
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Channeling Sahai ...

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P(x) = 2x^2 + 4x + 2
\]

Message? $P(1) = 1, P(2) = 4,$
Bad reception!

Send: (1,1), (2,4), (3,4), (4,7), (5,2), (6,0)

Recieve: (1,1) (2,4), (6,0)
Reconstruct?

Format: $(i, R(i))$. 

Lagrange or linear equations.

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P(6) = 2a_2 + 3a_1 + a_0 \equiv 0 \pmod{7}
\]

Channeling Sahai ...

\[
P(x) = 2x^2 + 4x + 2
\]

Message? $P(1) = 1, P(2) = 4, P(3) = 4$. 
You want to encode a secret consisting of 1,4,4.
Questions for Review

You want to encode a secret consisting of 1, 4, 4. How big should modulus be?
You want to encode a secret consisting of 1, 4, 4.

How big should modulus be?
Larger than 144
Questions for Review

You want to encode a secret consisting of 1,4,4.

How big should modulus be?
Larger than 144 and prime!

You want to send a message consisting of packets 1,4,2,3,0 through a noisy channel that loses 3 packets.

How big should modulus be?
Larger than 8 and prime!

The other constraint: arithmetic system can represent 0, 1, 2, 3, 4.

Send $n$ packets $b$-bit packets, with $k$ errors.

Modulus should be larger than $n + k$ and also larger than $2^b$. 
Questions for Review

You want to encode a secret consisting of 1, 4, 4.

How big should modulus be?
   Larger than 144 and prime!

Remember the secret, \( s = 144 \), must be one of the possible values.
Questions for Review

You want to encode a secret consisting of 1,4,4.

How big should modulus be?
  Larger than 144 and prime!

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Send $n$ packets $b$-bit packets, with $k$ errors.
Modulus should be larger than $n + k$ and also larger than $2^b$. 
Polynomials.
Polynomials.

- give Secret Sharing.
Polynomials.

- give Secret Sharing.
- give Erasure Codes.
Polynomials.

- give Secret Sharing.
- give Erasure Codes.

**Error Correction:**
Polynomials.

- give Secret Sharing.
- give Erasure Codes.

**Error Correction:**

Noisy Channel: corrupts $k$ packets. (rather than loss.)
Polynomials.

- give Secret Sharing.
- give Erasure Codes.

Error Correction:

Noisy Channel: corrupts $k$ packets. (rather than loss.)

Additional Challenge: Finding which packets are corrupt.
Error Correction

Satellite

GPS device
Error Correction

Satellite

GPS device

3 packet message.
Error Correction

Satellite

GPS device

3 packet message.

Corrupts 1 packets.
Error Correction

Satellite

3 packet message. Send 5.

Corrupts 1 packets.

GPS device
Error Correction

Satellite

3 packet message. Send 5.

Corrupts 1 packets.
Error Correction

Satellite

3 packet message. Send 5.

Corrupts 1 packets.
The Scheme.

**Problem:** Communicate \( n \) packets \( m_1, \ldots, m_n \) on noisy channel that corrupts \( \leq k \) packets.
The Scheme.

**Problem:** Communicate $n$ packets $m_1, \ldots, m_n$ on noisy channel that corrupts $\leq k$ packets.

**Reed-Solomon Code:**

1. Make a polynomial, $P(x)$ of degree $n - 1$, that encodes message.
   
   $P(1) = m_1, \ldots, P(n) = m_n$.
   
   Comment: could encode with packets as coefficients.

2. Send $P(1), \ldots, P(n+2k)$.

After noisy channel:

Receive values $R(1), \ldots, R(n+2k)$.

Properties:

(1) $P(i) = R(i)$ for at least $n+k$ points $i$,

(2) $P(x)$ is unique degree $n-1$ polynomial that contains $\geq n+k$ received points.
The Scheme.

**Problem:** Communicate $n$ packets $m_1, \ldots, m_n$ on noisy channel that corrupts $\leq k$ packets.

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2. Send $P(1), \ldots, P(n + 2k)$.
The Scheme.

**Problem:** Communicate \( n \) packets \( m_1, \ldots, m_n \) on noisy channel that corrupts \( \leq k \) packets.

**Reed-Solomon Code:**

1. Make a polynomial, \( P(x) \) of degree \( n - 1 \), that encodes message.
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   - Comment: could encode with packets as coefficients.

2. Send \( P(1), \ldots, P(n+2k) \).

**After noisy channel:** Recieve values \( R(1), \ldots, R(n+2k) \).
The Scheme.

**Problem:** Communicate $n$ packets $m_1, \ldots, m_n$ on noisy channel that corrupts $\leq k$ packets.

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**After noisy channel:** Recieve values $R(1), \ldots, R(n + 2k)$.

**Properties:**

1. $P(i) = R(i)$ for at least $n + k$ points $i$, 
The Scheme.

**Problem:** Communicate $n$ packets $m_1, \ldots, m_n$ on noisy channel that corrupts $\leq k$ packets.

**Reed-Solomon Code:**

1. Make a polynomial, $P(x)$ of degree $n - 1$, that encodes message.
   - $P(1) = m_1, \ldots, P(n) = m_n$.
   - Comment: could encode with packets as coefficients.

2. Send $P(1), \ldots, P(n + 2k)$.

**After noisy channel:** Recieve values $R(1), \ldots, R(n + 2k)$.

**Properties:**

1. $P(i) = R(i)$ for at least $n + k$ points $i$,
2. $P(x)$ is unique degree $n - 1$ polynomial
The Scheme.

**Problem:** Communicate $n$ packets $m_1, \ldots, m_n$ on noisy channel that corrupts $\leq k$ packets.

**Reed-Solomon Code:**

1. Make a polynomial, $P(x)$ of degree $n - 1$, that encodes message.
   
   $\quad \Box \quad P(1) = m_1, \ldots, P(n) = m_n.$
   
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**Properties:**

(1) $P(i) = R(i)$ for at least $n+k$ points $i$,

(2) $P(x)$ is unique degree $n-1$ polynomial that contains $\geq n+k$ received points.
Properties: proof.

\[ P(x) \]: degree \( n - 1 \) polynomial.
Properties: proof.

\[ P(x) : \text{degree } n - 1 \text{ polynomial.} \]
Send \( P(1), \ldots, P(n + 2k) \)
Properties: proof.

\( P(x) \): degree \( n - 1 \) polynomial.
Send \( P(1), \ldots, P(n + 2k) \)
Receive \( R(1), \ldots, R(n + 2k) \)
Properties: proof.

\[ P(x) \]: degree \( n - 1 \) polynomial.
Send \( P(1), \ldots, P(n + 2k) \)
Receive \( R(1), \ldots, R(n + 2k) \)
At most \( k \) i’s where \( P(i) \neq R(i) \).
Properties: proof.

\( P(x) \): degree \( n - 1 \) polynomial.
Send \( P(1), \ldots, P(n + 2k) \)
Receive \( R(1), \ldots, R(n + 2k) \)
At most \( k \) i’s where \( P(i) \neq R(i) \).

Properties:

1. \( P(i) = R(i) \) for at least \( n + k \) points \( i \),

   \( \geq P \) – Holes.

   Points contained by both: \( \geq n \).

   \( = \Rightarrow Q(i) = P(i) \) at \( n \) points.

   \( = \Rightarrow Q(x) = P(x) \).
Properties: proof.

\[ P(x): \text{degree } n - 1 \text{ polynomial.} \]
Send \( P(1), \ldots, P(n + 2k) \)
Receive \( R(1), \ldots, R(n + 2k) \)
At most \( k \) 'i's where \( P(i) \neq R(i) \).

Properties:

(1) \( P(i) = R(i) \) for at least \( n + k \) points \( i \),
(2) \( P(x) \) is unique degree \( n - 1 \) polynomial

Proof:

(1) Sure.
(2) Degree \( n - 1 \) polynomial \( Q(x) \) consistent with \( n + k \) points.
\( Q(x) \) agrees with \( R(i) \), \( n + k \) times.
\( P(x) \) agrees with \( R(i) \), \( n + k \) times.
Total points contained by both: \( 2n + 2k \).
Total points to choose from: \( n + 2k \).
Holes. Points contained by both: \( \geq n \).
\( \geq P - H \) Collisions.
\( = \Rightarrow Q(i) = P(i) \) at \( n \) points.
\( = \Rightarrow Q(x) = P(x) \).
Properties: proof.

\[ P(x) : \text{degree } n - 1 \text{ polynomial.} \]
Send \[ P(1), \ldots, P(n + 2k) \]
Receive \[ R(1), \ldots, R(n + 2k) \]
At most \( k \) i’s where \( P(i) \neq R(i) \).

Properties:
(1) \( P(i) = R(i) \) for at least \( n + k \) points \( i \),
(2) \( P(x) \) is unique degree \( n - 1 \) polynomial that contains \( \geq n + k \) received points.
Properties: proof.

\( P(x) \): degree \( n-1 \) polynomial.
Send \( P(1), \ldots, P(n+2k) \)
Receive \( R(1), \ldots, R(n+2k) \)
At most \( k \) \( i \)'s where \( P(i) \neq R(i) \).

Properties:
(1) \( P(i) = R(i) \) for at least \( n+k \) points \( i \),
(2) \( P(x) \) is unique degree \( n-1 \) polynomial that contains \( \geq n+k \) received points.

Proof:
Properties: proof.

$P(x)$: degree $n - 1$ polynomial.
Send \( P(1), \ldots, P(n + 2k) \)
Receive \( R(1), \ldots, R(n + 2k) \)
At most $k$ i’s where $P(i) \neq R(i)$.

Properties:
(1) $P(i) = R(i)$ for at least $n + k$ points $i$,
(2) $P(x)$ is unique degree $n - 1$ polynomial
that contains $\geq n + k$ received points.

Proof:
(1) Sure.
Properties: proof.

\( P(x) \): degree \( n - 1 \) polynomial.
Send \( P(1), \ldots, P(n+2k) \)
Receive \( R(1), \ldots, R(n+2k) \)
At most \( k \) i's where \( P(i) \neq R(i) \).

Properties:
(1) \( P(i) = R(i) \) for at least \( n + k \) points \( i \),
(2) \( P(x) \) is unique degree \( n - 1 \) polynomial
that contains \( \geq n + k \) received points.

Proof:
(1) Sure. Only \( k \) corruptions.
Properties: proof.

\(P(x)\): degree \(n-1\) polynomial.
Send \(P(1), \ldots, P(n+2k)\)
Receive \(R(1), \ldots, R(n+2k)\)
At most \(k\) \(i\)’s where \(P(i) \neq R(i)\).

Properties:
(1) \(P(i) = R(i)\) for at least \(n+k\) points \(i\),
(2) \(P(x)\) is unique degree \(n-1\) polynomial
that contains \(\geq n+k\) received points.

Proof:
(1) Sure. Only \(k\) corruptions.
(2) Degree \(n-1\) polynomial \(Q(x)\) consistent with \(n+k\) points.
Properties: proof.

\( P(x) \): degree \( n - 1 \) polynomial.
Send \( P(1), \ldots, P(n+2k) \)
Receive \( R(1), \ldots, R(n+2k) \)
At most \( k \) i’s where \( P(i) \neq R(i) \).

Properties:
(1) \( P(i) = R(i) \) for at least \( n+k \) points \( i \),
(2) \( P(x) \) is unique degree \( n - 1 \) polynomial
    that contains \( \geq n+k \) received points.

Proof:
(1) Sure. Only \( k \) corruptions.
(2) Degree \( n - 1 \) polynomial \( Q(x) \) consistent with \( n+k \) points.
    \( Q(x) \) agrees with \( R(i), n+k \) times.
Properties: proof.

\( P(x) \): degree \( n - 1 \) polynomial.
Send \( P(1), \ldots, P(n+2k) \)
Receive \( R(1), \ldots, R(n+2k) \)
At most \( k \) i’s where \( P(i) \neq R(i) \).

**Properties:**
(1) \( P(i) = R(i) \) for at least \( n+k \) points \( i \),
(2) \( P(x) \) is unique degree \( n - 1 \) polynomial
    that contains \( \geq n+k \) received points.

**Proof:**
(1) Sure. Only \( k \) corruptions.
(2) Degree \( n - 1 \) polynomial \( Q(x) \) consistent with \( n+k \) points.
    \( Q(x) \) agrees with \( R(i) \), \( n+k \) times.
    \( P(x) \) agrees with \( R(i) \), \( n+k \) times.
Properties: proof.

$P(x)$: degree $n - 1$ polynomial.
Send $P(1), \ldots, P(n + 2k)$
Receive $R(1), \ldots, R(n + 2k)$
At most $k$ i's where $P(i) \neq R(i)$.

Properties:
(1) $P(i) = R(i)$ for at least $n + k$ points $i$,
(2) $P(x)$ is unique degree $n - 1$ polynomial
that contains $\geq n + k$ received points.

Proof:
(1) Sure. Only $k$ corruptions.
(2) Degree $n - 1$ polynomial $Q(x)$ consistent with $n + k$ points.
$Q(x)$ agrees with $R(i)$, $n + k$ times.
$P(x)$ agrees with $R(i)$, $n + k$ times.
Total points contained by both: $2n + 2k$. 
Properties: proof.

\( P(x) \): degree \( n - 1 \) polynomial.
Send \( P(1), \ldots, P(n + 2k) \)
Receive \( R(1), \ldots, R(n + 2k) \)
At most \( k \) i’s where \( P(i) \neq R(i) \).

Properties:
(1) \( P(i) = R(i) \) for at least \( n + k \) points \( i \),
(2) \( P(x) \) is unique degree \( n - 1 \) polynomial
  that contains \( \geq n + k \) received points.

Proof:
(1) Sure. Only \( k \) corruptions.
(2) Degree \( n - 1 \) polynomial \( Q(x) \) consistent with \( n + k \) points.
  \( Q(x) \) agrees with \( R(i) \), \( n + k \) times.
  \( P(x) \) agrees with \( R(i) \), \( n + k \) times.
  Total points contained by both: \( 2n + 2k \). Pigeons.
Properties: proof.

\[ P(x) \]: degree \( n - 1 \) polynomial.
Send \( P(1), \ldots, P(n + 2k) \)
Receive \( R(1), \ldots, R(n + 2k) \)
At most \( k \) \( i \)'s where \( P(i) \neq R(i) \).

Properties:
(1) \( P(i) = R(i) \) for at least \( n + k \) points \( i \),
(2) \( P(x) \) is unique degree \( n - 1 \) polynomial
    that contains \( \geq n + k \) received points.

Proof:
(1) Sure. Only \( k \) corruptions.
(2) Degree \( n - 1 \) polynomial \( Q(x) \) consistent with \( n + k \) points.
    \( Q(x) \) agrees with \( R(i), n + k \) times.
    \( P(x) \) agrees with \( R(i), n + k \) times.
    Total points contained by both: \( 2n + 2k \). \( P \) Pigeons.
    Total points to choose from \( : n + 2k \).
Properties: proof.

$P(x)$: degree $n - 1$ polynomial.
Send $P(1), \ldots, P(n+2k)$
Receive $R(1), \ldots, R(n+2k)$
At most $k$ $i$’s where $P(i) \neq R(i)$.

Properties:
(1) $P(i) = R(i)$ for at least $n+k$ points $i$,
(2) $P(x)$ is unique degree $n - 1$ polynomial
    that contains $\geq n+k$ received points.

Proof:
(1) Sure. Only $k$ corruptions.
(2) Degree $n - 1$ polynomial $Q(x)$ consistent with $n+k$ points.
    $Q(x)$ agrees with $R(i)$, $n+k$ times.
    $P(x)$ agrees with $R(i)$, $n+k$ times.
    Total points contained by both: $2n+2k$. $P$ Pigeons.
    Total points to choose from : $n+2k$. $H$ Holes.
Properties: proof.

\( P(x) \): degree \( n - 1 \) polynomial.
Send \( P(1), \ldots, P(n + 2k) \)
Receive \( R(1), \ldots, R(n + 2k) \)
At most \( k \) i's where \( P(i) \neq R(i) \).

Properties:
(1) \( P(i) = R(i) \) for at least \( n + k \) points \( i \),
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    that contains \( \geq n + k \) received points.

Proof:
(1) Sure. Only \( k \) corruptions.
(2) Degree \( n - 1 \) polynomial \( Q(x) \) consistent with \( n + k \) points.
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    Total points contained by both: \( 2n + 2k \). \( P \) Pigeons.
    Total points to choose from : \( n + 2k \). \( H \) Holes.
    Points contained by both : \( \geq n \).
Properties: proof.

\( P(x) \): degree \( n - 1 \) polynomial.
Send \( P(1), \ldots, P(n + 2k) \)
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At most \( k \) \( i \)'s where \( P(i) \neq R(i) \).

Properties:
(1) \( P(i) = R(i) \) for at least \( n + k \) points \( i \),
(2) \( P(x) \) is unique degree \( n - 1 \) polynomial
that contains \( \geq n + k \) received points.

Proof:
(1) Sure. Only \( k \) corruptions.
(2) Degree \( n - 1 \) polynomial \( Q(x) \) consistent with \( n + k \) points.
\( Q(x) \) agrees with \( R(i) \), \( n + k \) times.
\( P(x) \) agrees with \( R(i) \), \( n + k \) times.
Total points contained by both: \( 2n + 2k \).  \( P \quad \text{Pigeons.} \)
Total points to choose from : \( n + 2k \).  \( H \quad \text{Holes.} \)
Points contained by both : \( \geq n \).  \( \geq P - H \quad \text{Collisions.} \)
\( \implies Q(i) = P(i) \) at \( n \) points.
Properties: proof.

\( P(x) \): degree \( n - 1 \) polynomial.
Send \( P(1), \ldots, P(n+2k) \)
Receive \( R(1), \ldots, R(n+2k) \)
At most \( k \) i’s where \( P(i) \neq R(i) \).

Properties:
(1) \( P(i) = R(i) \) for at least \( n+k \) points \( i \),
(2) \( P(x) \) is unique degree \( n - 1 \) polynomial
    that contains \( \geq n+k \) received points.

Proof:
(1) Sure. Only \( k \) corruptions.
(2) Degree \( n - 1 \) polynomial \( Q(x) \) consistent with \( n+k \) points.
    \( Q(x) \) agrees with \( R(i) \), \( n+k \) times.
    \( P(x) \) agrees with \( R(i) \), \( n+k \) times.
    Total points contained by both: \( 2n+2k \). \( P \) \ Pigeons.
    Total points to choose from : \( n+2k \). \( H \) \ Holes.
    Points contained by both : \( \geq n. \geq P - H \) \ Collisions.
\implies Q(i) = P(i) \ at \ n \ points.
\implies Q(x) = P(x).
Properties: proof.

\( P(x) \): degree \( n - 1 \) polynomial.
Send \( P(1), \ldots, P(n + 2k) \)
Receive \( R(1), \ldots, R(n + 2k) \)
At most \( k \) 's where \( P(i) \neq R(i) \).

Properties:
(1) \( P(i) = R(i) \) for at least \( n + k \) points \( i \),
(2) \( P(x) \) is unique degree \( n - 1 \) polynomial
    that contains \( \geq n + k \) received points.

Proof:
(1) Sure. Only \( k \) corruptions.
(2) Degree \( n - 1 \) polynomial \( Q(x) \) consistent with \( n + k \) points.
  \( Q(x) \) agrees with \( R(i) \), \( n + k \) times.
  \( P(x) \) agrees with \( R(i) \), \( n + k \) times.
  Total points contained by both: \( 2n + 2k \).  \( P \) Pigeons.
  Total points to choose from : \( n + 2k \).  \( H \) Holes.
  Points contained by both : \( \geq n \).  \( \geq P - H \) Collisions.
  \( \implies Q(i) = P(i) \) at \( n \) points.
  \( \implies Q(x) = P(x) \).
Example.

Message: 3, 0, 6.
Example.

Message: 3, 0, 6.

Reed Solomon Code: \( P(x) = x^2 + x + 1 \) (mod 7) has \( P(1) = 3, P(2) = 0, P(3) = 6 \) modulo 7.
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Send: \( P(1) = 3, P(2) = 0, P(3) = 6, P(4) = 0, P(5) = 3 \).
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(Aside: Message in plain text!)
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(Aside: Message in plain text!)

Receive \( R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3. \)

\( P(i) = R(i) \) for \( n + k = 3 + 1 = 4 \) points.
Slow solution.

**Brute Force:**
For each subset of $n + k$ points
Slow solution.

**Brute Force:**
For each subset of \( n + k \) points
  - Fit degree \( n - 1 \) polynomial, \( Q(x) \), to \( n \) of them.
Slow solution.

**Brute Force:**
For each subset of $n + k$ points
  - Fit degree $n - 1$ polynomial, $Q(x)$, to $n$ of them.
  - Check if consistent with $n + k$ of the total points.
Slow solution.

**Brute Force:**
For each subset of $n + k$ points
- Fit degree $n-1$ polynomial, $Q(x)$, to $n$ of them.
- Check if consistent with $n+k$ of the total points.
- If yes, output $Q(x)$.  

For subset of $n+k$ pts where $R(i) = P(i)$, method will reconstruct $P(x)$!

For any subset of $n+k$ pts,
1. there is unique degree $n-1$ polynomial $Q(x)$ that fits $n$ of them
2. and where $Q(x)$ is consistent with $n+k$ points $\Rightarrow P(x) = Q(x)$.

Reconstructs $P(x)$ and only $P(x)$!!
**Brute Force:**
For each subset of $n+k$ points
   Fit degree $n-1$ polynomial, $Q(x)$, to $n$ of them.
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Reconstructs $P(x)$ and only $P(x)$!!
Example.

Received $R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3$
Example.

Received $R(1) = 3$, $R(2) = 1$, $R(3) = 6$, $R(4) = 0$, $R(5) = 3$

Find $P(x) = p_2 x^2 + p_1 x + p_0$ that contains $n + k = 3 + 1$ points.
Example.

Received $R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3$

Find $P(x) = p_2 x^2 + p_1 x + p_0$ that contains $n + k = 3 + 1$ points.

All equations...

\[
\begin{align*}
p_2 + p_1 + p_0 & \equiv 3 \pmod{7} \\
4p_2 + 2p_1 + p_0 & \equiv 1 \pmod{7} \\
2p_2 + 3p_1 + p_0 & \equiv 6 \pmod{7} \\
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Assume point 1 is wrong
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Assume point 1 is wrong and solve..
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Assume point 1 is wrong and solve.. \textbf{no consistent solution!}
Example.

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\end{align*}
\]

Assume point 1 is wrong and solve.. \textbf{no consistent solution!}
Assume point 2 is wrong
Example.

Received $R(1) = 3$, $R(2) = 1$, $R(3) = 6$, $R(4) = 0$, $R(5) = 3$

Find $P(x) = p_2 x^2 + p_1 x + p_0$ that contains $n + k = 3 + 1$ points.

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Assume point 1 is wrong and solve.. no consistent solution!
Assume point 2 is wrong and solve...
Example.

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Assume point 1 is wrong and solve..\textcolor{red}{no consistent solution!}
Assume point 2 is wrong and solve...\textcolor{green}{consistent solution!}
In general,

\[ P(x) = p_{n-1}x^{n-1} + \cdots p_0 \] and receive \( R(1), \ldots R(m = n + 2k) \).
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\[ p_{n-1}2^{n-1} + \cdots p_0 \equiv R(2) \pmod{p} \]

Error!!... Where???

Could be anywhere!!...

...so try everywhere.

Runtime: \((n + 2k)^k\) possibilities.

Something like \(\left(\frac{n}{k}\right)^k\)...Exponential in \(k\)!

How do we find where the bad packets are efficiently?!?!?
In general, 

\[ P(x) = p_{n-1}x^{n-1} + \cdots + p_0 \] and receive \( R(1), \ldots, R(m = n + 2k) \).

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**Runtime:** \( \binom{n+2k}{k} \) possibilities.
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**Runtime:** \( \binom{n+2k}{k} \) possibilities.

Something like \( (n/k)^k \) ...Exponential in \( k \)!.
In general,

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Something like \( (n/k)^k \) ...Exponential in \( k \).

How do we find where the bad packets are efficiently?!?!?!
Ditty...

Oh where, Oh where

Oh where, Oh where
Ditty...

Oh where, Oh where has my little dog gone?

Oh where, oh where can he be
With his ears cut short
And his tail cut long
Oh where, oh where can he be?

Oh where, Oh where have my packets gone...
Oh where, oh where do they not fit.
With the polynomial well put
But the channel a bit wrong
Where, oh where do we look?
Oh where, Oh where
has my little dog gone?
Oh where, oh where can he be
Ditty...

Oh where, Oh where
has my little dog gone?
Oh where, oh where can he be

With his ears cut short
Ditty...

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With the polynomial well put
But the channel a bit wrong
Where, oh where do we look?
Where oh where can my bad packets be?

\[(p_{n-1} + \cdots p_0) \equiv R(1) \pmod{p}\]
Where oh where can my bad packets be?

\[(p_{n-1} + \cdots p_0) \equiv R(1) \pmod{p}\]

\[(p_{n-1}2^{n-1} + \cdots p_0) \equiv R(2) \pmod{p}\]

\[\vdots\]

\[(p_{n-1}(m)^{n-1} + \cdots p_0) \equiv R(n+2k) \pmod{p}\]
Where oh where can my **bad packets** be?

\[
(p_{n-1} + \cdots + p_0) \equiv R(1) \pmod{p}
\]
\[
(p_{n-1}2^{n-1} + \cdots + p_0) \equiv R(2) \pmod{p}
\]
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\vdots
\]
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**Idea:** Multiply equation \(i\) by 0 if and only if \(P(i) \neq R(i)\).
Where oh where can my bad packets be?

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**Idea:** Multiply equation \( i \) by 0 if and only if \( P(i) \neq R(i) \). Zero times anything is zero!!!!!
Where oh where can my **bad** packets be?

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**Idea:** Multiply equation \(i\) by 0 if and only if \(P(i) \neq R(i)\). Zero times anything is zero!!!! My love is won.
Where oh where can my **bad** packets be?

\[(p_{n-1} + \cdots + p_0) \equiv R(1) \pmod{p}\]

\[0 \times (p_{n-1}2^{n-1} + \cdots + p_0) \equiv R(2) \pmod{p}\]

\[\vdots\]

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**Idea:** Multiply equation \(i\) by 0 if and only if \(P(i) \neq R(i)\).

Zero times anything is zero!!!!! My love is won.

All equations satisfied!!!!!
Where oh where can my bad packets be?

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(p_{n-1}2^{n-1} + \cdots + p_0) & \equiv R(2) \pmod{p} \\
& \vdots \\
(p_{n-1}(m)^{n-1} + \cdots + p_0) & \equiv R(n+2k) \pmod{p}
\end{align*}
\]

**Idea:** Multiply equation \(i\) by 0 if and only if \(P(i) \neq R(i)\). Zero times anything is zero!!!!! My love is won. All equations satisfied!!!!!

But which equations should we multiply by 0?
Where oh where can my **bad packets** be?

\[
\begin{align*}
(p_{n-1} + \cdots + p_0) & \equiv R(1) \pmod{p} \\
(p_{n-1}2^{n-1} + \cdots + p_0) & \equiv R(2) \pmod{p} \\
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But which equations should we multiply by 0? Where oh where...??

We will use a polynomial!!! That we don’t know.
Where oh where can my bad packets be?

\[(p_{n-1} + \cdots + p_0) \equiv R(1) \pmod{p}\]
\[(p_{n-1}2^{n-1} + \cdots + p_0) \equiv R(2) \pmod{p}\]
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Zero times anything is zero!!!! My love is won.
All equations satisfied!!!!!

But which equations should we multiply by 0? Where oh where...??

We will use a polynomial!!! That we don’t know. But can find!
Where oh where can my **bad packets** be?

\[
(p_{n-1} + \cdots p_0) \equiv R(1) \pmod{p}
\]
\[
(p_{n-1}2^{n-1} + \cdots p_0) \equiv R(2) \pmod{p}
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\[\vdots\]
\[
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But which equations should we multiply by 0? Where oh where...??

We will use a polynomial!!! That we don’t know. But can find!

Errors at points \( e_1, \ldots, e_k \). (In diagram above, \( e_1 = 2 \).)
Where oh where can my bad packets be?

\[
(p_{n-1} + \cdots + p_0) \equiv R(1) \pmod{p} \\
(p_{n-1}2^{n-1} + \cdots + p_0) \equiv R(2) \pmod{p} \\
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Errors at points \(e_1, \ldots, e_k\). (In diagram above, \(e_1 = 2\).)

**Error locator polynomial:** \(E(x) = (x - e_1)\)
Where oh where can my bad packets be?

\[(p_{n-1} + \cdots + p_0) \equiv R(1) \pmod{p}\]
\[(p_{n-1}2^{n-1} + \cdots + p_0) \equiv R(2) \pmod{p}\]
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Errors at points \(e_1, \ldots, e_k\). (In diagram above, \(e_1 = 2\).)

**Error locator polynomial:** \(E(x) = (x - e_1)(x - e_2)\)
Where oh where can my bad packets be?

\[(p_{n-1} + \cdots + p_0) \equiv R(1) \pmod{p}\]
\[(p_{n-1}2^{n-1} + \cdots + p_0) \equiv R(2) \pmod{p}\]
\[\vdots\]
\[(p_{n-1}(m)^{n-1} + \cdots + p_0) \equiv R(n+2k) \pmod{p}\]

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Where oh where can my bad packets be?

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\begin{align*}
(p_{n-1} + \cdots p_0) & \equiv R(1) \pmod{p} \\
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\begin{align*}
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Errors at points \( e_1, \ldots, e_k \). (In diagram above, \( e_1 = 2 \).)

**Error locator polynomial:** \( E(x) = (x - e_1)(x - e_2)\ldots(x - e_k) \).
\( E(i) = 0 \) if and only if \( e_j = i \) for some \( j \)
Where oh where can my bad packets be?

\[
E(1)(p_{n-1} + \cdots p_0) \equiv R(1)E(1) \pmod{p}
\]
\[
E(2)(p_{n-1}2^{n-1} + \cdots p_0) \equiv R(2)E(2) \pmod{p}
\]
\[
\vdots
\]
\[
E(m)(p_{n-1}(m)^{n-1} + \cdots p_0) \equiv R(n+2k)E(m) \pmod{p}
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\(E(i) = 0\) if and only if \(e_j = i\) for some \(j\)

Multiply equations by \(E(\cdot)\).
Where oh where can my bad packets be?

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E(1)(p_{n-1} + \cdots p_0) \equiv R(1)E(1) \pmod{p}
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\[
E(2)(p_{n-1}2^{n-1} + \cdots p_0) \equiv R(2)E(2) \pmod{p}
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Errors at points \(e_1, \ldots, e_k\). (In diagram above, \(e_1 = 2\).)

**Error locator polynomial:** \(E(x) = (x - e_1)(x - e_2) \cdots (x - e_k)\).

\(E(i) = 0\) if and only if \(e_j = i\) for some \(j\)

Multiply equations by \(E(\cdot)\). (Above \(E(x) = (x-2)\).)
Where oh where can my **bad packets** be?

$$E(1)(p_{n-1} + \cdots + p_0) \equiv R(1)E(1) \pmod{p}$$
$$E(2)(p_{n-1}2^{n-1} + \cdots + p_0) \equiv R(2)E(2) \pmod{p}$$
$$\vdots$$
$$E(m)(p_{n-1}(m)^{n-1} + \cdots + p_0) \equiv R(n+2k)E(m) \pmod{p}$$

**Idea:** Multiply equation $i$ by 0 if and only if $P(i) \neq R(i)$.
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Errors at points $e_1, \ldots, e_k$. (In diagram above, $e_1 = 2$.)

**Error locator polynomial:** $E(x) = (x - e_1)(x - e_2)\ldots(x - e_k)$.

$E(i) = 0$ if and only if $e_j = i$ for some $j$

Multiply equations by $E(\cdot)$. (Above $E(x) = (x-2)$.)

All equations satisfied!!!
Example.

Received \( R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3 \)
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Find \( P(x) = p_2 x^2 + p_1 x + p_0 \) that contains \( n + k = 3 + 1 \) points.
Example.

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Find \( P(x) = p_2 x^2 + p_1 x + p_0 \) that contains \( n + k = 3 + 1 \) points.

Plugin points...

\[
\begin{align*}
(p_2 + p_1 + p_0) & \equiv (3) \pmod{7} \\
(4p_2 + 2p_1 + p_0) & \equiv (1) \pmod{7} \\
(2p_2 + 3p_1 + p_0) & \equiv (6) \pmod{7} \\
(2p_2 + 4p_1 + p_0) & \equiv (0) \pmod{7} \\
(4p_2 + 5p_1 + p_0) & \equiv (3) \pmod{7}
\end{align*}
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Plugin points...

\[
(p_2 + p_1 + p_0) \equiv 3 \pmod{7}
\]
\[
(4p_2 + 2p_1 + p_0) \equiv 1 \pmod{7}
\]
\[
(2p_2 + 3p_1 + p_0) \equiv 6 \pmod{7}
\]
\[
(2p_2 + 4p_1 + p_0) \equiv 0 \pmod{7}
\]
\[
(4p_2 + 5p_1 + p_0) \equiv 3 \pmod{7}
\]

Error locator polynomial: $(x - 2)$. 

Example.

Received $R(1) = 3$, $R(2) = 1$, $R(3) = 6$, $R(4) = 0$, $R(5) = 3$

Find $P(x) = p_2 x^2 + p_1 x + p_0$ that contains $n + k = 3 + 1$ points.

Plugin points...

\[
(1 - 2)(p_2 + p_1 + p_0) \equiv (3)(1 - 2) \pmod{7}
\]

\[
(2 - 2)(4p_2 + 2p_1 + p_0) \equiv (1)(2 - 2) \pmod{7}
\]

\[
(3 - 2)(2p_2 + 3p_1 + p_0) \equiv (6)(3 - 2) \pmod{7}
\]

\[
(4 - 2)(2p_2 + 4p_1 + p_0) \equiv (0)(4 - 2) \pmod{7}
\]

\[
(5 - 2)(4p_2 + 5p_1 + p_0) \equiv (3)(5 - 2) \pmod{7}
\]

Error locator polynomial: $(x - 2)$.

Multiply equation $i$ by $(i - 2)$. 
Example.

Received $R(1) = 3$, $R(2) = 1$, $R(3) = 6$, $R(4) = 0$, $R(5) = 3$

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\]
\[
(5 - 2)(4p_2 + 5p_1 + p_0) \equiv (3)(5 - 2) \pmod{7}
\]

Error locator polynomial: $(x - 2)$.

Multiply equation $i$ by $(i - 2)$. All equations satisfied!
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Received $R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3$.

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Error locator polynomial: $(x - 2)$.

Multiply equation $i$ by $(i - 2)$. All equations satisfied!

But don’t know error locator polynomial!
Example.

Received $R(1) = 3$, $R(2) = 1$, $R(3) = 6$, $R(4) = 0$, $R(5) = 3$

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$$(1 - 2)(p_2 + p_1 + p_0) \equiv (3)(1 - 2) \pmod{7}$$

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Error locator polynomial: $(x - 2)$.

Multiply equation $i$ by $(i - 2)$. All equations satisfied!

But don’t know error locator polynomial! Do know form:
Example.

Received $R(1) = 3$, $R(2) = 1$, $R(3) = 6$, $R(4) = 0$, $R(5) = 3$

Find $P(x) = p_2 x^2 + p_1 x + p_0$ that contains $n + k = 3 + 1$ points.

Plugin points...

\[
(1 - 2)(p_2 + p_1 + p_0) \equiv (3)(1 - 2) \pmod{7}
\]

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(2 - 2)(4p_2 + 2p_1 + p_0) \equiv (1)(2 - 2) \pmod{7}
\]

\[
(3 - 2)(2p_2 + 3p_1 + p_0) \equiv (6)(3 - 2) \pmod{7}
\]

\[
(4 - 2)(2p_2 + 4p_1 + p_0) \equiv (0)(4 - 2) \pmod{7}
\]

\[
(5 - 2)(4p_2 + 5p_1 + p_0) \equiv (3)(5 - 2) \pmod{7}
\]

Error locator polynomial: $(x - 2)$.

Multiply equation $i$ by $(i - 2)$. All equations satisfied!

But don’t know error locator polynomial! Do know form: $(x - e)$. 
Example.

Received $R(1) = 3$, $R(2) = 1$, $R(3) = 6$, $R(4) = 0$, $R(5) = 3$

Find $P(x) = p_2x^2 + p_1x + p_0$ that contains $n + k = 3 + 1$ points.

Plugin points...

\[
(1 - e)(p_2 + p_1 + p_0) \equiv (3)(1 - e) \pmod{7}
\]
\[
(2 - e)(4p_2 + 2p_1 + p_0) \equiv (1)(2 - e) \pmod{7}
\]
\[
(3 - e)(2p_2 + 3p_1 + p_0) \equiv (3)(3 - e) \pmod{7}
\]
\[
(4 - e)(2p_2 + 4p_1 + p_0) \equiv (0)(4 - e) \pmod{7}
\]
\[
(5 - e)(4p_2 + 5p_1 + p_0) \equiv (3)(5 - e) \pmod{7}
\]

Error locator polynomial: $(x - 2)$.

Multiply equation $i$ by $(i - 2)$. All equations satisfied!

But don’t know error locator polynomial! Do know form: $(x - e)$. 

Example.

Received \( R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3 \)

Find \( P(x) = p_2 x^2 + p_1 x + p_0 \) that contains \( n + k = 3 + 1 \) points.

Plugin points...

\[
(1 - e)(p_2 + p_1 + p_0) \equiv (3)(1 - e) \pmod 7
\]
\[
(2 - e)(4p_2 + 2p_1 + p_0) \equiv (1)(2 - e) \pmod 7
\]
\[
(3 - e)(2p_2 + 3p_1 + p_0) \equiv (3)(3 - e) \pmod 7
\]
\[
(4 - e)(2p_2 + 4p_1 + p_0) \equiv (0)(4 - e) \pmod 7
\]
\[
(5 - e)(4p_2 + 5p_1 + p_0) \equiv (3)(5 - e) \pmod 7
\]

Error locator polynomial: \( (x - 2) \).

Multiply equation \( i \) by \( (i - 2) \). All equations satisfied!

But don’t know error locator polynomial! Do know form: \( (x - e) \).

4 unknowns \( (p_0, p_1, p_2 \text{ and } e) \),
Example.

Received $R(1) = 3$, $R(2) = 1$, $R(3) = 6$, $R(4) = 0$, $R(5) = 3$

Find $P(x) = p_2 x^2 + p_1 x + p_0$ that contains $n+k = 3+1$ points.

Plugin points...

\[(1 - e)(p_2 + p_1 + p_0) \equiv (3)(1 - e) \pmod{7}\]
\[(2 - e)(4p_2 + 2p_1 + p_0) \equiv (1)(2 - e) \pmod{7}\]
\[(3 - e)(2p_2 + 3p_1 + p_0) \equiv (3)(3 - e) \pmod{7}\]
\[(4 - e)(2p_2 + 4p_1 + p_0) \equiv (0)(4 - e) \pmod{7}\]
\[(5 - e)(4p_2 + 5p_1 + p_0) \equiv (3)(5 - e) \pmod{7}\]

Error locator polynomial: $(x - 2)$.

Multiply equation $i$ by $(i - 2)$. All equations satisfied!

But don’t know error locator polynomial! Do know form: $(x - e)$.

4 unknowns ($p_0, p_1, p_2$ and $e$), 5 nonlinear equations.
.. turn their heads each day,

\[
(p_{n-1} + \cdots + p_0) \equiv R(1) \pmod{p} \\
\vdots
\\n(p_{n-1}i^{n-1} + \cdots + p_0) \equiv R(i) \pmod{p} \\
\vdots
\\n(p_{n-1}(n+2k)^{n-1} + \cdots + p_0) \equiv R(m) \pmod{p}
\]
..turn their heads each day,

\[
E(1)(p_{n-1} + \cdots p_0) \equiv R(1)E(1) \pmod{p}
\]
\[
\vdots
\]
\[
E(i)(p_{n-1}i^{n-1} + \cdots p_0) \equiv R(i)E(i) \pmod{p}
\]
\[
\vdots
\]
\[
E(m)(p_{n-1}(n+2k)^{n-1} + \cdots p_0) \equiv R(m)E(m) \pmod{p}
\]

...so satisfied, I’m on my way.
. . . turn their heads each day,

\[ E(1)(p_{n-1} + \cdots p_0) \equiv R(1)E(1) \pmod{p} \]
\[ \vdots \]
\[ E(i)(p_{n-1}i^{n-1} + \cdots p_0) \equiv R(i)E(i) \pmod{p} \]
\[ \vdots \]
\[ E(m)(p_{n-1}(n+2k)^{n-1} + \cdots p_0) \equiv R(m)E(m) \pmod{p} \]

...so satisfied, I’m on my way.

\[ m = n + 2k \] satisfied equations,
..turn their heads each day,

\[
E(1)(p_{n-1} + \cdots p_0) \equiv R(1)E(1) \pmod{p}
\]
\[
\vdots
\]
\[
E(i)(p_{n-1}i^{n-1} + \cdots p_0) \equiv R(i)E(i) \pmod{p}
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E(m)(p_{n-1}(n+2k)^{n-1} + \cdots p_0) \equiv R(m)E(m) \pmod{p}
\]

...so satisfied, I’m on my way.

\[m = n + 2k \text{ satisfied equations, } n + k \text{ unknowns.}\]
..turn their heads each day,

\[ E(1)(p_{n-1} + \cdots p_0) \equiv R(1)E(1) \pmod{p} \]

\[
\vdots
\]

\[ E(i)(p_{n-1}i^{n-1} + \cdots p_0) \equiv R(i)E(i) \pmod{p} \]

\[
\vdots
\]

\[ E(m)(p_{n-1}(n+2k)^{n-1} + \cdots p_0) \equiv R(m)E(m) \pmod{p} \]

...so satisfied, I’m on my way.

\[ m = n + 2k \text{ satisfied equations, } n + k \text{ unknowns. But nonlinear!} \]
..turn their heads each day,

\[
E(1)(p_{n-1} + \cdots p_0) \equiv R(1)E(1) \pmod{p}
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\[\vdots\]
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\[\vdots\]
\[
E(m)(p_{n-1}(n+2k)^{n-1} + \cdots p_0) \equiv R(m)E(m) \pmod{p}
\]

...so satisfied, I’m on my way.

\[m = n + 2k\] satisfied equations, \(n + k\) unknowns. But nonlinear!

Let \(Q(x) = E(x)P(x) = a_{n+k-1}x^{n+k-1} + \cdots a_0\).
\[ E(1)(p_{n-1} + \cdots p_0) \equiv R(1)E(1) \pmod{p} \]
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\[ E(i)(p_{n-1} i^{n-1} + \cdots p_0) \equiv R(i)E(i) \pmod{p} \]
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Equations:
\[ Q(i) = R(i)E(i). \]
..turn their heads each day,

\[
E(1)(\rho_{n-1} + \cdots \rho_0) \equiv R(1)E(1) \pmod{p}
\]
\[
\vdots
\]
\[
E(i)(\rho_{n-1}i^{n-1} + \cdots \rho_0) \equiv R(i)E(i) \pmod{p}
\]
\[
\vdots
\]
\[
E(m)(\rho_{n-1}(n+2k)^{n-1} + \cdots \rho_0) \equiv R(m)E(m) \pmod{p}
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\]

...so satisfied, I’m on my way.

\[m = n + 2k \text{ satisfied equations, } n + k \text{ unknowns. But nonlinear!}\]

Let \(Q(x) = E(x)P(x) = a_{n+k-1}x^{n+k-1} + \cdots a_0\).

Equations:

\[Q(i) = R(i)E(i)\]

and linear in \(a_i\) and coefficients of \(E(x)\)!
Finding $Q(x)$ and $E(x)$?
Finding $Q(x)$ and $E(x)$?

- $E(x)$ has degree $k$
Finding $Q(x)$ and $E(x)$?

- $E(x)$ has degree $k$ ...

\[ E(x) = x^k + b_{k-1}x^{k-1} \cdots b_0. \]
Finding $Q(x)$ and $E(x)$?

- $E(x)$ has degree $k$ ...

$$E(x) = x^k + b_{k-1}x^{k-1} \cdots b_0.$$  

$$\implies k \text{ (unknown) coefficients.}$$
Finding $Q(x)$ and $E(x)$?

$E(x)$ has degree $k$ ...

$$E(x) = x^k + b_{k-1} x^{k-1} \cdots b_0.$$  

$\implies k$ (unknown) coefficients. Leading coefficient is 1.
Finding $Q(x)$ and $E(x)$?

- $E(x)$ has degree $k$ ...

$$E(x) = x^k + b_{k-1} x^{k-1} \cdots b_0.$$  

$\implies k$ (unknown) coefficients. Leading coefficient is 1.

- $Q(x) = P(x)E(x)$ has degree $n + k - 1$
Finding $Q(x)$ and $E(x)$?

- $E(x)$ has degree $k$ ...

\[ E(x) = x^k + b_{k-1}x^{k-1} \cdots b_0. \]

\[ \Rightarrow \quad k \text{ (unknown) coefficients. Leading coefficient is 1.} \]

- $Q(x) = P(x)E(x)$ has degree $n + k - 1$ ...

\[ Q(x) = a_{n+k-1}x^{n+k-1} + a_{n+k-2}x^{n+k-2} + \cdots a_0 \]
Finding $Q(x)$ and $E(x)$?

- $E(x)$ has degree $k$ ...

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$\implies n + k$ (unknown) coefficients.
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- $E(x)$ has degree $k$ ...

$$E(x) = x^k + b_{k-1} x^{k-1} \cdots b_0.$$  

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$\implies n + k$ (unknown) coefficients.

Number of unknown coefficients:
Finding \( Q(x) \) and \( E(x) \)?

- \( E(x) \) has degree \( k \) ...
  \[
  E(x) = x^k + b_{k-1}x^{k-1} \cdots b_0. 
  \]
  \[\implies \] \( k \) (unknown) coefficients. Leading coefficient is 1.

- \( Q(x) = P(x)E(x) \) has degree \( n+k-1 \) ...
  \[
  Q(x) = a_{n+k-1}x^{n+k-1} + a_{n+k-2}x^{n+k-2} + \cdots a_0
  \]
  \[\implies \] \( n+k \) (unknown) coefficients.

Number of unknown coefficients: \( n+2k \).
Solving for $Q(x)$ and $E(x)$...

For all points $1, \ldots, i, n + 2k = m$,

$$Q(i) = R(i)E(i) \pmod{p}$$
Solving for $Q(x)$ and $E(x)$...

For all points $1, \ldots, i, n + 2k = m,$

$$Q(i) = R(i)E(i) \pmod{p}$$

Gives $n + 2k$ linear equations.
Solving for $Q(x)$ and $E(x)$...

For all points $1, \ldots, i, n + 2k = m$,

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Gives $n + 2k$ linear equations.

$$a_{n+k-1} + \cdots a_0 \equiv R(1)(1 + b_{k-1}\cdots b_0) \pmod{p}$$
Solving for $Q(x)$ and $E(x)$...

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$$a_{n+k-1}(2)^{n+k-1} + \cdots + a_0 \equiv R(2)(((2)^k + b_{k-1}(2)^{k-1} \cdots b_0) \pmod{p}$$

$$\vdots$$
Solving for $Q(x)$ and $E(x)$...

For all points $1, \ldots, i, n+2k = m$,

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$$\vdots$$

$$a_{n+k-1}(m)^{n+k-1} + \ldots a_0 \equiv R(m)((m)^k + b_{k-1}(m)^{k-1} \ldots b_0) \pmod{p}$$
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..and $n + 2k$ unknown coefficients of $Q(x)$ and $E(x)$!
Solving for $Q(x)$ and $E(x)$...

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..and $n+2k$ unknown coefficients of $Q(x)$ and $E(x)$!

Solve for coefficients of $Q(x)$ and $E(x)$.
Solving for $Q(x)$ and $E(x)$...and $P(x)$

For all points $1,\ldots,i,n+2k = m$,

$$Q(i) = R(i)E(i) \pmod{p}$$

Gives $n+2k$ linear equations.

$$a_{n+k-1} + \ldots a_0 \equiv R(1)(1 + b_{k-1} \ldots b_0) \pmod{p}$$

$$a_{n+k-1}(2)^{n+k-1} + \ldots a_0 \equiv R(2)((2)^k + b_{k-1}(2)^{k-1} \ldots b_0) \pmod{p}$$

$$\vdots$$

$$a_{n+k-1}(m)^{n+k-1} + \ldots a_0 \equiv R(m)((m)^k + b_{k-1}(m)^{k-1} \ldots b_0) \pmod{p}$$

..and $n+2k$ unknown coefficients of $Q(x)$ and $E(x)$!

Solve for coefficients of $Q(x)$ and $E(x)$.

Find $P(x) = Q(x)/E(x)$. 
Solving for $Q(x)$ and $E(x)$...and $P(x)$

For all points $1, \ldots, i, n+2k = m$,

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$$\vdots$$

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Solving for $Q(x)$ and $E(x)$...and $P(x)$

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Solving for $Q(x)$ and $E(x)$...and $P(x)$

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..and $n + 2k$ unknown coefficients of $Q(x)$ and $E(x)$!

Solve for coefficients of $Q(x)$ and $E(x)$.

Find $P(x) = Q(x)/E(x)$. 
Example.

Received $R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3$
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$Q(x) = E(x)P(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$
Example.

Received $R(1) = 3$, $R(2) = 1$, $R(3) = 6$, $R(4) = 0$, $R(5) = 3$

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\[
a_3 + a_2 + a_1 + a_0 \equiv 3(1 - b_0) \pmod{7}
\]
Example.

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$$a_3 + a_2 + a_1 + a_0 \equiv 3(1 - b_0) \pmod{7}$$

$$a_3 + 4a_2 + 2a_1 + a_0 \equiv 1(2 - b_0) \pmod{7}$$
Example.

Received $R(1) = 3$, $R(2) = 1$, $R(3) = 6$, $R(4) = 0$, $R(5) = 3$

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$Q(i) = R(i)E(i).$

\[
\begin{align*}
  a_3 + a_2 + a_1 + a_0 & \equiv 3(1 - b_0) \pmod{7} \\
  a_3 + 4a_2 + 2a_1 + a_0 & \equiv 1(2 - b_0) \pmod{7} \\
  6a_3 + 2a_2 + 3a_1 + a_0 & \equiv 6(3 - b_0) \pmod{7} \\
  a_3 + 2a_2 + 4a_1 + a_0 & \equiv 0(4 - b_0) \pmod{7} \\
  6a_3 + 4a_2 + 5a_1 + a_0 & \equiv 3(5 - b_0) \pmod{7}
\end{align*}
\]
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Received \( R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3 \)

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    a_3 + 2a_2 + 4a_1 + a_0 & \equiv 0(4 - b_0) \pmod{7} \\
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\]

\( a_3 = 1, a_2 = 6, a_1 = 6, a_0 = 5 \) and \( b_0 = 2. \)
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\[
\begin{align*}
    a_3 + a_2 + a_1 + a_0 & \equiv 3(1 - b_0) \quad (\text{mod } 7) \\
    a_3 + 4a_2 + 2a_1 + a_0 & \equiv 1(2 - b_0) \quad (\text{mod } 7) \\
    6a_3 + 2a_2 + 3a_1 + a_0 & \equiv 6(3 - b_0) \quad (\text{mod } 7) \\
    a_3 + 2a_2 + 4a_1 + a_0 & \equiv 0(4 - b_0) \quad (\text{mod } 7) \\
    6a_3 + 4a_2 + 5a_1 + a_0 & \equiv 3(5 - b_0) \quad (\text{mod } 7)
\end{align*}
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$a_3 = 1, a_2 = 6, a_1 = 6, a_0 = 5$ and $b_0 = 2$.

$Q(x) = x^3 + 6x^2 + 6x + 5$. 
Example.

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$Q(x) = x^3 + 6x^2 + 6x + 5$.

$E(x) = x - 2$. 
Example: finishing up.

\[ Q(x) = x^3 + 6x^2 + 6x + 5. \]
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Example: finishing up.

\[ Q(x) = x^3 + 6x^2 + 6x + 5. \]
\[ E(x) = x - 2. \]

\[
\begin{array}{r}\begin{array}{c}
 x - 2 \end{array} & \begin{array}{r}
 x^3 + 6x^2 + 6x + 5
\end{array} \\
\hline
 x^3 - 2x^2
\end{array}
\]
\[
\begin{array}{r}
 x^2 + 6x + 5 \\
 x^2 - 2x
\end{array}
\]
\[
\begin{array}{r}
 6x + 5 \\
 6x - 12
\end{array}
\]
\[
\begin{array}{r}
 0 \\
 -2
\end{array}
\]

\[ P(x) = x^2 + x + 1 \]

Message is

\[ P(1) = 3, \quad P(2) = 0, \quad P(3) = 6. \]

What is \( x - 2 \)?

Except at \( x = 2 \)?

Hole there?
Example: finishing up.

\[ Q(x) = x^3 + 6x^2 + 6x + 5. \]

\[ E(x) = x - 2. \]

\[
\begin{array}{c}
1 \\
\hline
x - 2 \\
\hline
\end{array}
\]

\[
\begin{array}{c}
x^3 + 6x^2 + 6x + 5 \\
\hline
x^3 - 2x^2 \\
\hline
1x^2 + 6x + 5 \\
\hline
1x^2 - 2x \\
\hline
x + 5 \\
\hline
0
\end{array}
\]

\[ P(x) = x^2 + x + 1. \]

Message is \( P(1) = 3, P(2) = 0, P(3) = 6. \)

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\[
\begin{array}{c|ccccc}
& 1 & x^2 & & & \\
\hline
x - 2 & x^3 & + & 6x^2 & + & 6x + 5 \\
\hline
& x^3 & - & 2x^2 & & \\
\hline
& 1 & x^2 & + & 6x + 5
\end{array}
\]

Message is
\[ P(1) = 3, \quad P(2) = 0, \quad P(3) = 6. \]
What is \( x - 2 \)?

\[ \text{Except at } x = 2? \]
Hole there?
Example: finishing up.

\[ Q(x) = x^3 + 6x^2 + 6x + 5. \]
\[ E(x) = x - 2. \]

\[
\begin{array}{r}
1 & x^2 & + & 1 & x \\
\hline
x - 2 & ) & x^3 & + & 6x^2 & + & 6x & + & 5 \\
& & x^3 & - & 2x^2 \\
& & \hline
& & 1 & x^2 & + & 6x & + & 5 \\
& & 1 & x^2 & - & 2x \\
\end{array}
\]

Message is

\[ P(x) = x^2 + x + 1 \]

\[ P(1) = 3, \quad P(2) = 0, \quad P(3) = 6. \]

What is \( x - 2 \) except at \( x = 2 \)?
Example: finishing up.

\[ Q(x) = x^3 + 6x^2 + 6x + 5. \]
\[ E(x) = x - 2. \]

\[
\begin{array}{r}
1 \ x^2 + 1 \ x \\
\hline
\x - 2 \ ) \ x^3 + 6 \ x^2 + 6 \ x + 5 \\
\hline
\x^3 - 2 \ x^2 \\
\hline
1 \ x^2 + 6 \ x + 5 \\
\hline
1 \ x^2 - 2 \ x \\
\hline
\x + 5
\end{array}
\]

Message is

\[ P(1) = 3, \quad P(2) = 0, \quad P(3) = 6. \]

What is \( x - 2 \) at \( x = 2 \)?

Hole there?
Example: finishing up.

\[ Q(x) = x^3 + 6x^2 + 6x + 5. \]
\[ E(x) = x - 2. \]

\[ \begin{array}{r}
1 \ x^2 + 1 \ x + 1 \\
\hline
x - 2 \ ) \ x^3 + 6 \ x^2 + 6 \ x + 5 \\
x^3 - 2 \ x^2 \\
\hline
1 \ x^2 + 6 \ x + 5 \\
1 \ x^2 - 2 \ x \\
\hline
x + 5 \\
x - 2
\end{array} \]
Example: finishing up.

\[ Q(x) = x^3 + 6x^2 + 6x + 5. \]
\[ E(x) = x - 2. \]

\[
\begin{array}{c}
    1 \ x^2 + 1 \ x + 1 \\
\hline
    x - 2 \big) x^3 + 6 \ x^2 + 6 \ x + 5 \\
    x^3 - 2 \ x^2 \\
\hline
    1 \ x^2 + 6 \ x + 5 \\
    1 \ x^2 - 2 \ x \\
\hline
    x + 5 \\
\end{array}
\]

Message is

\[ P(x) = x^2 + x + 1 \]
\[ P(1) = 3, \quad P(2) = 0, \quad P(3) = 6. \]

What is \( x - 2 \)?

Except at \( x = 2? \)
Example: finishing up.

\[ Q(x) = x^3 + 6x^2 + 6x + 5. \]
\[ E(x) = x - 2. \]

\[
\begin{array}{r}
1 \ x^2 + 1 \ x + 1 \\
\hline
x - 2 \ 
\end{array}
\]

\[
\begin{array}{r}
\ x^3 + 6 \ x^2 + 6 \ x + 5 \\
\ x^3 - 2 \ x^2 \\
\hline
\ x^2 + 6 \ x + 5 \\
\ 1 \ x^2 - 2 \ x \\
\hline
\ x + 5 \\
\ x - 2 \\
\hline
\ 0
\end{array}
\]

\[ P(x) = x^2 + x + 1 \]
Example: finishing up.

\[ Q(x) = x^3 + 6x^2 + 6x + 5. \]
\[ E(x) = x - 2. \]

\[
\begin{array}{r}
1 & x^2 & + & 1 & x & + & 1 \\
\hline
x - 2 & ) & x^3 & + & 6 & x^2 & + & 6 & x & + & 5 \\
& x^3 & - & 2 & x^2 & & & & & & & \hline
& & 1 & x^2 & + & 6 & x & + & 5
\end{array}
\]

\[
\begin{array}{r}
1 & x^2 & - & 2 & x \\
\hline
& & 1 & x^2 & - & 2 & x \\
& & & x & + & 5 \\
& & & x & - & 2 \\
& & & & & & 0
\end{array}
\]

\[ P(x) = x^2 + x + 1 \]
Message is \( P(1) = 3, P(2) = 0, P(3) = 6. \)
Example: finishing up.

\[ Q(x) = x^3 + 6x^2 + 6x + 5. \]
\[ E(x) = x - 2. \]

\[
\begin{array}{c|ccccc}
& x^3 & + & 6x^2 & + & 6x & + & 5 \\
\hline
x - 2 & - & \hline
\end{array}
\]

\[
\begin{array}{c|cccc}
& x^3 & + & 6x^2 & + & 6x & + & 5 \\
\hline
1x^2 & + & 1x & + & 1 & \hline
\end{array}
\]

\[
\begin{array}{c|cccc}
x - 2 & - & \hline
1x^2 & + & 6x & + & 5 & \hline
1x^2 & - & 2x & \hline
\end{array}
\]

\[
\begin{array}{c|cc}
x + 5 & \hline
x - 2 & \hline
\end{array}
\]

\[ P(x) = x^2 + x + 1 \]
Message is \[ P(1) = 3, P(2) = 0, P(3) = 6. \]
What is \[ \frac{x-2}{x-2} \]?
Example: finishing up.

\[ Q(x) = x^3 + 6x^2 + 6x + 5. \]
\[ E(x) = x - 2. \]

\[
\begin{array}{r}
 1 & x^2 & + & 1 & x & + & 1 \\
\hline
 x - 2 & x^3 & + & 6 & x^2 & + & 6 & x & + & 5 \\
 & x^3 & - & 2 & x^2 \\
\hline
 & 1 & x^2 & + & 6 & x & + & 5 \\
 & 1 & x^2 & - & 2 & x \\
\hline
 & x & + & 5 \\
 & x - 2 \\
\hline
 & 0
\end{array}
\]

\[ P(x) = x^2 + x + 1 \]
Message is \( P(1) = 3, P(2) = 0, P(3) = 6. \)
What is \( \frac{x-2}{x-2} \)? 1
Example: finishing up.

\[ Q(x) = x^3 + 6x^2 + 6x + 5. \]
\[ E(x) = x - 2. \]

\[
\begin{array}{c}
1 \ x^2 + 1 \ x + 1 \\
\hline
x - 2 ) \ x^3 + 6 \ x^2 + 6 \ x + 5 \\
\hline
x^3 - 2 \ x^2 \\
\hline
1 \ x^2 + 6 \ x + 5 \\
1 \ x^2 - 2 \ x \\
\hline
x + 5 \\
x - 2 \\
\hline
0
\end{array}
\]

\[ P(x) = x^2 + x + 1 \]
Message is \( P(1) = 3, P(2) = 0, P(3) = 6. \)

What is \( \frac{x-2}{x-2} \)?  1
   Except at \( x = 2 \)?
Example: finishing up.

\[ Q(x) = x^3 + 6x^2 + 6x + 5. \]
\[ E(x) = x - 2. \]

\[
\begin{array}{c}
1 \ x^2 + 1 \ x + 1 \\
\hline
x - 2) x^3 + 6x^2 + 6x + 5 \\
x^3 - 2x^2 \\
\hline
1x^2 + 6x + 5 \\
1x^2 - 2x \\
\hline
x + 5 \\
\hline
0
\end{array}
\]

\[ P(x) = x^2 + x + 1 \]
Message is \( P(1) = 3, P(2) = 0, P(3) = 6. \)

What is \( \frac{x-2}{x-2} \)? 1
Except at \( x = 2 \)? Hole there?
Error Correction: Berlekamp-Welsh

Message: $m_1, \ldots, m_n$.

**Sender:**

1. Form degree $n - 1$ polynomial $P(x)$ where $P(i) = m_i$.
2. Send $P(1), \ldots, P(n + 2k)$.

**Receiver:**

1. Receive $R(1), \ldots, R(n + 2k)$.
2. Solve $n + 2k$ equations, $Q(i) = E(i)R(i)$ to find $Q(x) = E(x)P(x)$ and $E(x)$.
3. Compute $P(x) = Q(x)/E(x)$.
4. Compute $P(1), \ldots, P(n)$. 
Check your understanding.

You have error locator polynomial!
Check your understanding.

You have error locator polynomial!
Where oh where have my packets gone wrong?
Check your understanding.

You have error locator polynomial!
Where oh where have my packets gone wrong?
Factor?
Check your understanding.

You have error locator polynomial!
Where oh where have my packets gone wrong?
Factor? Sure.
Check your understanding.

You have error locator polynomial!

Where oh where have my packets gone wrong?

Factor? Sure.

Check all values?
Check your understanding.

You have error locator polynomial!
Where oh where have my packets gone wrong?
Factor? Sure.
Check all values? Sure.
Check your understanding.

You have error locator polynomial!
Where oh where have my packets gone wrong?
Factor? Sure.
Check all values? Sure.
Check your understanding.

You have error locator polynomial!
Where oh where have my packets gone wrong?
Factor? Sure.
Check all values? Sure.
Efficiency?
You have error locator polynomial!
Where oh where have my packets gone *wrong*?
Factor? Sure.
Check all values? Sure.
Efficiency? Sure.
You have error locator polynomial!
Where oh where have my packets gone wrong?
Factor? Sure.
Check all values? Sure.
Efficiency? Sure. Only $n + 2k$ values.
Check your understanding.

You have error locator polynomial!
Where oh where have my packets gone wrong?
Factor? Sure.
Check all values? Sure.
Efficiency? Sure. Only $n + 2k$ values.
See where it is 0.
Hmmm...

Is there one and only one $P(x)$ from Berlekamp-Welsh procedure?
Is there one and only one $P(x)$ from Berlekamp-Welsh procedure?

**Existence:** there is a $P(x)$ and $E(x)$ that satisfy equations.
Unique solution for $P(x)$

**Uniqueness:** any solution $Q'(x)$ and $E'(x)$ have

$$\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x).$$  \hspace{1cm} (1)
Unique solution for $P(x)$

**Uniqueness:** any solution $Q'(x)$ and $E'(x)$ have

\[ \frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x). \quad (1) \]

**Proof:**
Unique solution for $P(x)$

**Uniqueness:** any solution $Q'(x)$ and $E'(x)$ have

$$\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x).$$  \hspace{1cm} (1)

**Proof:**

We claim
Unique solution for $P(x)$

**Uniqueness:** any solution $Q'(x)$ and $E'(x)$ have

$$\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x).$$  \hspace{1cm} (1)

**Proof:**

We claim

$$Q'(x)E(x) = Q(x)E'(x) \text{ on } n + 2k \text{ values of } x.$$  \hspace{1cm} (2)
Unique solution for $P(x)$

**Uniqueness:** any solution $Q'(x)$ and $E'(x)$ have

\[
\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x).
\]  

(1)

**Proof:**
We claim

\[
Q'(x)E(x) = Q(x)E'(x) \text{ on } n+2k \text{ values of } x.
\]  

(2)

Equation 2 implies 1:
Unique solution for \( P(x) \)

**Uniqueness:** any solution \( Q'(x) \) and \( E'(x) \) have

\[
\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x).
\]

(1)

**Proof:**
We claim

\[
Q'(x)E(x) = Q(x)E'(x) \text{ on } n+2k \text{ values of } x.
\]

(2)

Equation 2 implies 1:

\( Q'(x)E(x) \) and \( Q(x)E'(x) \) are degree \( n+2k-1 \)
Unique solution for $P(x)$

**Uniqueness:** any solution $Q'(x)$ and $E'(x)$ have

$$\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x).$$  \hspace{1cm} (1)

**Proof:**

We claim

$$Q'(x)E(x) = Q(x)E'(x)$$ on $n + 2k$ values of $x$.  \hspace{1cm} (2)

Equation 2 implies 1:

$Q'(x)E(x)$ and $Q(x)E'(x)$ are degree $n + 2k - 1$
and agree on $n + 2k$ points
Unique solution for $P(x)$

**Uniqueness:** any solution $Q'(x)$ and $E'(x)$ have

\[
\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x).
\]  

(1)

**Proof:**

We claim

\[
Q'(x)E(x) = Q(x)E'(x) \text{ on } n + 2k \text{ values of } x. 
\]  

(2)

Equation 2 implies 1:

$Q'(x)E(x)$ and $Q(x)E'(x)$ are degree $n + 2k - 1$
and agree on $n + 2k$ points

$E(x)$ and $E'(x)$ have at most $k$ zeros each.
Unique solution for $P(x)$

**Uniqueness**: any solution $Q'(x)$ and $E'(x)$ have

$$\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x).$$  \hspace{1cm} (1)

**Proof:**

We claim

$$Q'(x)E(x) = Q(x)E'(x) \text{ on } n+2k \text{ values of } x. \hspace{1cm} (2)$$

Equation 2 implies 1:

$Q'(x)E(x)$ and $Q(x)E'(x)$ are degree $n+2k-1$

and agree on $n+2k$ points

$E(x)$ and $E'(x)$ have at most $k$ zeros each.

Can cross divide at $n$ points.
Unique solution for $P(x)$

**Uniqueness:** any solution $Q'(x)$ and $E'(x)$ have

$$\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x).$$  \hspace{1cm} (1)

**Proof:**

We claim

$$Q'(x)E(x) = Q(x)E'(x)$$ on $n + 2k$ values of $x$. \hspace{1cm} (2)

Equation 2 implies 1:

$Q'(x)E(x)$ and $Q(x)E'(x)$ are degree $n + 2k - 1$

and agree on $n + 2k$ points

$E(x)$ and $E'(x)$ have at most $k$ zeros each.

Can cross divide at $n$ points.

$$\implies \frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)}$$ equal on $n$ points.
Unique solution for $P(x)$

**Uniqueness:** any solution $Q'(x)$ and $E'(x)$ have

$$
\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x).
$$

(1)

**Proof:**
We claim

$$Q'(x)E(x) = Q(x)E'(x) \text{ on } n + 2k \text{ values of } x. \quad (2)$$

Equation 2 implies 1:

$Q'(x)E(x)$ and $Q(x)E'(x)$ are degree $n + 2k - 1$

and agree on $n + 2k$ points

$E(x)$ and $E'(x)$ have at most $k$ zeros each.

Can cross divide at $n$ points.

$$\Rightarrow \quad \frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} \text{ equal on } n \text{ points.}$$

Both degree $\leq n$
Unique solution for $P(x)$

**Uniqueness**: any solution $Q'(x)$ and $E'(x)$ have

\[
\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x).
\] (1)

**Proof:**
We claim

\[Q'(x)E(x) = Q(x)E'(x)\] on $n + 2k$ values of $x$. (2)

Equation 2 implies 1:

$Q'(x)E(x)$ and $Q(x)E'(x)$ are degree $n + 2k - 1$ and agree on $n + 2k$ points.

$E(x)$ and $E'(x)$ have at most $k$ zeros each.

Can cross divide at $n$ points.

\[\Rightarrow \frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)}\] equal on $n$ points.

Both degree $\leq n \Rightarrow$ Same polynomial!
Unique solution for $P(x)$

**Uniqueness:** any solution $Q'(x)$ and $E'(x)$ have

$$
\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x).
$$

(1)

**Proof:**

We claim

$$
Q'(x)E(x) = Q(x)E'(x) \text{ on } n+2k \text{ values of } x.
$$

(2)

Equation 2 implies 1:

$Q'(x)E(x)$ and $Q(x)E'(x)$ are degree $n+2k-1$

and agree on $n+2k$ points

$E(x)$ and $E'(x)$ have at most $k$ zeros each.

Can cross divide at $n$ points.

$$
\Longrightarrow \frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} \text{ equal on } n \text{ points}.
$$

Both degree $\leq n \Rightarrow$ Same polynomial!
Fact: $Q'(x)E(x) = Q(x)E'(x)$ on $n + 2k$ values of $x$. 
Last bit.

Fact: \( Q'(x)E(x) = Q(x)E'(x) \) on \( n + 2k \) values of \( x \).

Proof:

Cross multiplying gives equality in fact for these points. Points to polynomials, have to deal with zeros! Example: dealing with \( x^2 - 2x - 2 \) at \( x = 2 \).
Fact: \( Q'(x)E(x) = Q(x)E'(x) \) on \( n + 2k \) values of \( x \).

Proof: Construction implies that

\[ Q'(i)E(i) = Q(i)E'(i) \]

for \( i \in \{1, \ldots, n+2k\} \).

If \( E(i) = 0 \), then \( Q(i) = 0 \).

If \( E'(i) = 0 \), then \( Q'(i) = 0 \).

\( \implies \)

\[ Q(i)E'(i) = Q'(i)E(i) \]

holds when \( E(i) \) or \( E'(i) \) are zero.

When \( E'(i) \) and \( E(i) \) are not zero

\[ Q'(i)E'(i) = Q(i)E(i) = R(i) \]

Cross multiplying gives equality in fact for these points.

Points to polynomials, have to deal with zeros!

Example: dealing with \( x^2 - 2x - 2 \) at \( x = 2 \).
Fact: \( Q'(x)E(x) = Q(x)E'(x) \) on \( n + 2k \) values of \( x \).

Proof: Construction implies that

\[
Q(i) = R(i)E(i)
\]
\[
Q'(i) = R(i)E'(i)
\]

Cross multiplying gives equality in fact for these points.

Points to polynomials, have to deal with zeros!

Example: dealing with \( x^2 - 2x - 2 \) at \( x = 2 \).
Fact: \( Q'(x)E(x) = Q(x)E'(x) \) on \( n+2k \) values of \( x \).

Proof: Construction implies that

\[
Q(i) = R(i)E(i) \\
Q'(i) = R(i)E'(i)
\]

for \( i \in \{1, \ldots n+2k\} \).
Factor: $Q'(x)E(x) = Q(x)E'(x)$ on $n + 2k$ values of $x$.

Proof: Construction implies that

$$Q(i) = R(i)E(i)$$
$$Q'(i) = R(i)E'(i)$$

for $i \in \{1, \ldots n+2k\}$.

If $E(i) = 0$, then $Q(i) = 0$. 

Cross multiplying gives equality in fact for these points.

Points to polynomials, have to deal with zeros!
Fact: \( Q'(x)E(x) = Q(x)E'(x) \) on \( n + 2k \) values of \( x \).

Proof: Construction implies that

\[
Q(i) = R(i)E(i) \\
Q'(i) = R(i)E'(i)
\]

for \( i \in \{1, \ldots n + 2k\} \).

If \( E(i) = 0 \), then \( Q(i) = 0 \). If \( E'(i) = 0 \), then \( Q'(i) = 0 \).
Fact: \( Q'(x)E(x) = Q(x)E'(x) \) on \( n + 2k \) values of \( x \).

Proof: Construction implies that

\[
Q(i) = R(i)E(i) \\
Q'(i) = R(i)E'(i)
\]

for \( i \in \{1, \ldots, n + 2k\} \).

If \( E(i) = 0 \), then \( Q(i) = 0 \). If \( E'(i) = 0 \), then \( Q'(i) = 0 \).

\[\Rightarrow Q(i)E'(i) = Q'(i)E(i) \text{ holds when } E(i) \text{ or } E'(i) \text{ are zero.}\]
Fact: $Q'(x)E(x) = Q(x)E'(x)$ on $n + 2k$ values of $x$.

Proof: Construction implies that 

$$Q(i) = R(i)E(i)$$

$$Q'(i) = R(i)E'(i)$$

for $i \in \{1, \ldots, n + 2k\}$. 

If $E(i) = 0$, then $Q(i) = 0$. If $E'(i) = 0$, then $Q'(i) = 0$. 

$\implies Q(i)E'(i) = Q'(i)E(i)$ holds when $E(i)$ or $E'(i)$ are zero.

When $E'(i)$ and $E(i)$ are not zero
Fact: \( Q'(x)E(x) = Q(x)E'(x) \) on \( n + 2k \) values of \( x \).

Proof: Construction implies that

\[
Q(i) = R(i)E(i) \\
Q'(i) = R(i)E'(i)
\]

for \( i \in \{1, \ldots n + 2k\} \).

If \( E(i) = 0 \), then \( Q(i) = 0 \). If \( E'(i) = 0 \), then \( Q'(i) = 0 \).

\[ \implies Q(i)E'(i) = Q'(i)E(i) \]

holds when \( E(i) \) or \( E'(i) \) are zero.

When \( E'(i) \) and \( E(i) \) are not zero

\[
\frac{Q'(i)}{E'(i)} = \frac{Q(i)}{E(i)} = R(i).
\]
Fact: \( Q'(x)E(x) = Q(x)E'(x) \) on \( n+2k \) values of \( x \).

Proof: Construction implies that

\[
Q(i) = R(i)E(i) \\
Q'(i) = R(i)E'(i)
\]

for \( i \in \{1, \ldots n + 2k\} \).

If \( E(i) = 0 \), then \( Q(i) = 0 \). If \( E'(i) = 0 \), then \( Q'(i) = 0 \).

\[\implies Q(i)E'(i) = Q'(i)E(i) \text{ holds when } E(i) \text{ or } E'(i) \text{ are zero.}\]

When \( E'(i) \) and \( E(i) \) are not zero

\[
\frac{Q'(i)}{E'(i)} = \frac{Q(i)}{E(i)} = R(i).
\]

Cross multiplying gives equality in fact for these points.
Fact: \( Q'(x)E(x) = Q(x)E'(x) \) on \( n + 2k \) values of \( x \).

Proof: Construction implies that

\[
Q(i) = R(i)E(i) \\
Q'(i) = R(i)E'(i)
\]

for \( i \in \{1, \ldots n + 2k\} \).

If \( E(i) = 0 \), then \( Q(i) = 0 \). If \( E'(i) = 0 \), then \( Q'(i) = 0 \).

\[ \Rightarrow \quad Q(i)E'(i) = Q'(i)E(i) \]

holds when \( E(i) \) or \( E'(i) \) are zero.

When \( E'(i) \) and \( E(i) \) are not zero

\[
\frac{Q'(i)}{E'(i)} = \frac{Q(i)}{E(i)} = R(i).
\]

Cross multiplying gives equality in fact for these points. \( \square \)
Fact: \( Q'(x)E(x) = Q(x)E'(x) \) on \( n+2k \) values of \( x \).

Proof: Construction implies that

\[
Q(i) = R(i)E(i) \\
Q'(i) = R(i)E'(i)
\]

for \( i \in \{1, \ldots n+2k\} \).

If \( E(i) = 0 \), then \( Q(i) = 0 \). If \( E'(i) = 0 \), then \( Q'(i) = 0 \).

\[ \implies Q(i)E'(i) = Q'(i)E(i) \]
holds when \( E(i) \) or \( E'(i) \) are zero.

When \( E'(i) \) and \( E(i) \) are not zero

\[
\frac{Q'(i)}{E'(i)} = \frac{Q(i)}{E(i)} = R(i).
\]

Cross multiplying gives equality in fact for these points.

Points to polynomials, have to deal with zeros!
Fact: \( Q'(x)E(x) = Q(x)E'(x) \) on \( n + 2k \) values of \( x \).

Proof: Construction implies that

\[
Q(i) = R(i)E(i) \\
Q'(i) = R(i)E'(i)
\]

for \( i \in \{1, \ldots n + 2k\} \).

If \( E(i) = 0 \), then \( Q(i) = 0 \). If \( E'(i) = 0 \), then \( Q'(i) = 0 \).

\[ \implies Q(i)E'(i) = Q'(i)E(i) \] holds when \( E(i) \) or \( E'(i) \) are zero.

When \( E'(i) \) and \( E(i) \) are not zero

\[
\frac{Q'(i)}{E'(i)} = \frac{Q(i)}{E(i)} = R(i).
\]

Cross multiplying gives equality in fact for these points.

Points to polynomials, have to deal with zeros!

Example: dealing with \( \frac{x-2}{x-2} \) at \( x = 2 \).
Berlekamp-Welsh algorithm decodes correctly when $k$ errors!
Summary. Error Correction.

Communicate $n$ packets, with $k$ erasures.
Summary. Error Correction.

Communicate $n$ packets, with $k$ erasures.

How many packets?
Summary. Error Correction.

Communicate $n$ packets, with $k$ erasures.

How many packets? $n + k$
Summary. Error Correction.

Communicate $n$ packets, with $k$ erasures.

How many packets? $n + k$
How to encode?
Summary. Error Correction.

Communicate \( n \) packets, with \( k \) erasures.

How many packets? \( n + k \)
How to encode? With polynomial, \( P(x) \).
Summary. Error Correction.

Communicate \( n \) packets, with \( k \) erasures.

How many packets? \( n + k \)
How to encode? With polynomial, \( P(x) \).
Of degree?
Summary. Error Correction.

Communicate $n$ packets, with $k$ erasures.

How many packets? $n + k$

How to encode? With polynomial, $P(x)$.

Of degree? $n – 1$
Communicate $n$ packets, with $k$ erasures.

How many packets? $n + k$

How to encode? With polynomial, $P(x)$.

Of degree? $n - 1$

Recover?
Summary. Error Correction.

Communicate $n$ packets, with $k$ erasures.

How many packets? $n + k$

How to encode? With polynomial, $P(x)$.

Of degree? $n - 1$

Recover? Reconstruct $P(x)$ with any $n$ points!
Summary. Error Correction.

Communicate $n$ packets, with $k$ erasures.

How many packets? $n + k$
How to encode? With polynomial, $P(x)$.
Of degree? $n - 1$
Recover? Reconstruct $P(x)$ with any $n$ points!

Communicate $n$ packets, with $k$ errors.
Summary. Error Correction.

Communicate $n$ packets, with $k$ erasures.
How many packets? $n + k$
How to encode? With polynomial, $P(x)$.
Of degree? $n - 1$
Recover? Reconstruct $P(x)$ with any $n$ points!

Communicate $n$ packets, with $k$ errors.
How many packets?
Summary. Error Correction.

Communicate $n$ packets, with $k$ erasures.

How many packets? $n + k$
How to encode? With polynomial, $P(x)$.
Of degree? $n - 1$
Recover? Reconstruct $P(x)$ with any $n$ points!

Communicate $n$ packets, with $k$ errors.

How many packets? $n + 2k$
Summary. Error Correction.

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- How many packets? $n + 2k$
- Why?
Summary. Error Correction.

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Why?

$k$ changes to make diff. messages overlap
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  - $k$ changes to make diff. messages overlap
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Summary. Error Correction.

Communicate \( n \) packets, with \( k \) erasures.

- How many packets? \( n + k \)
- How to encode? With polynomial, \( P(x) \).
- Of degree? \( n - 1 \)
- Recover? Reconstruct \( P(x) \) with any \( n \) points!

Communicate \( n \) packets, with \( k \) errors.

- How many packets? \( n + 2k \)
- Why?
  - \( k \) changes to make diff. messages overlap
- How to encode? With polynomial, \( P(x) \). Of degree?
Summary. Error Correction.

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How to encode? With polynomial, \( P(x) \).
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\( k \) changes to make diff. messages overlap
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Summary. Error Correction.

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Why?
  \( k \) changes to make diff. messages overlap
How to encode? With polynomial, \( P(x) \). Of degree? \( n - 1 \).
Recover?
Summary. Error Correction.

Communicate $n$ packets, with $k$ erasures.

- How many packets? $n + k$
- How to encode? With polynomial, $P(x)$.
- Of degree? $n - 1$
- Recover? Reconstruct $P(x)$ with any $n$ points!

Communicate $n$ packets, with $k$ errors.

- How many packets? $n + 2k$
- Why?
  - $k$ changes to make diff. messages overlap
- Recover?
  - Reconstruct error polynomial, $E(X)$, and $P(x)$!
Summary. Error Correction.

Communicate \( n \) packets, with \( k \) erasures.

How many packets? \( n + k \)
How to encode? With polynomial, \( P(x) \).
Of degree? \( n - 1 \)
Recover? Reconstruct \( P(x) \) with any \( n \) points!

Communicate \( n \) packets, with \( k \) errors.

How many packets? \( n + 2k \)
Why?
\( k \) changes to make diff. messages overlap
How to encode? With polynomial, \( P(x) \). Of degree? \( n - 1 \).
Recover?
Reconstruct error polynomial, \( E(X) \), and \( P(x) \)!
Nonlinear equations.
Summary. Error Correction.

Communicate \( n \) packets, with \( k \) erasures.

- How many packets? \( n + k \)
- How to encode? With polynomial, \( P(x) \).
- Of degree? \( n - 1 \)
- Recover? Reconstruct \( P(x) \) with any \( n \) points!

Communicate \( n \) packets, with \( k \) errors.

- How many packets? \( n + 2k \)
  - Why?
    - \( k \) changes to make diff. messages overlap
- How to encode? With polynomial, \( P(x) \). Of degree? \( n - 1 \).
- Recover?
  - Reconstruct error polynomial, \( E(X) \), and \( P(x) \)!
    - Nonlinear equations.
  - Reconstruct \( E(x) \) and \( Q(x) = E(x)P(x) \).
Summary. Error Correction.

Communicate $n$ packets, with $k$ erasures.

- How many packets? $n + k$
- How to encode? With polynomial, $P(x)$.
- Of degree? $n - 1$
- Recover? Reconstruct $P(x)$ with any $n$ points!

Communicate $n$ packets, with $k$ errors.

- How many packets? $n + 2k$
- Why?
  - $k$ changes to make diff. messages overlap
- Recover?
  - Reconstruct error polynomial, $E(X)$, and $P(x)$!
    - Nonlinear equations.
  - Reconstruct $E(x)$ and $Q(x) = E(x)P(x)$. Linear Equations.
Summary. Error Correction.

Communicate \( n \) packets, with \( k \) erasures.

- How many packets? \( n + k \)
- How to encode? With polynomial, \( P(x) \).
- Of degree? \( n - 1 \)
- Recover? Reconstruct \( P(x) \) with any \( n \) points!

Communicate \( n \) packets, with \( k \) errors.

- How many packets? \( n + 2k \)
- Why?
  - \( k \) changes to make different messages overlap
- How to encode? With polynomial, \( P(x) \). Of degree? \( n - 1 \).
- Recover?
  - Reconstruct error polynomial, \( E(X) \), and \( P(x) \)!
    - **Nonlinear equations.**
  - Reconstruct \( E(x) \) and \( Q(x) = E(x)P(x) \). Linear Equations.
  - Polynomial division!
Summary. Error Correction.

Communicate $n$ packets, with $k$ erasures.

- How many packets? $n + k$
- How to encode? With polynomial, $P(x)$.
- Of degree? $n - 1$
- Recover? Reconstruct $P(x)$ with any $n$ points!

Communicate $n$ packets, with $k$ errors.

- How many packets? $n + 2k$
- Why?
  - $k$ changes to make diff. messages overlap
- Recover?
  - Reconstruct error polynomial, $E(X)$, and $P(x)$!
    - Nonlinear equations.
  - Reconstruct $E(x)$ and $Q(x) = E(x)P(x)$. Linear Equations.
  - Polynomial division! $P(x) = Q(x)/E(x)$!
Communicate $n$ packets, with $k$ erasures.

How many packets? $n + k$
How to encode? With polynomial, $P(x)$.
Of degree? $n - 1$
Recover? Reconstruct $P(x)$ with any $n$ points!

Communicate $n$ packets, with $k$ errors.

How many packets? $n + 2k$
Why?
    $k$ changes to make diff. messages overlap
How to encode? With polynomial, $P(x)$. Of degree? $n - 1$.
Recover?
    Reconstruct error polynomial, $E(X)$, and $P(x)$!
    **Nonlinear equations.**
Reconstruct $E(x)$ and $Q(x) = E(x)P(x)$. Linear Equations. Polynomial division! $P(x) = Q(x)/E(x)$!

Reed-Solomon codes.
Summary. Error Correction.

Communicate \( n \) packets, with \( k \) erasures.

How many packets? \( n + k \)
How to encode? With polynomial, \( P(x) \).
Of degree? \( n - 1 \)
Recover? Reconstruct \( P(x) \) with any \( n \) points!

Communicate \( n \) packets, with \( k \) errors.

How many packets? \( n + 2k \)
Why?
\( k \) changes to make diff. messages overlap
How to encode? With polynomial, \( P(x) \). Of degree? \( n - 1 \).
Recover?
Reconstruct error polynomial, \( E(X) \), and \( P(x) \)!
   \textbf{Nonlinear equations.}
Reconstruct \( E(x) \) and \( Q(x) = E(x)P(x) \). Linear Equations.
Polynomial division! \( P(x) = Q(x)/E(x) \)!

Reed-Solomon codes. Welsh-Berlekamp Decoding.
Summary. Error Correction.

Communicate $n$ packets, with $k$ erasures.

How many packets? $n + k$
How to encode? With polynomial, $P(x)$.
Of degree? $n – 1$
Recover? Reconstruct $P(x)$ with any $n$ points!

Communicate $n$ packets, with $k$ errors.

How many packets? $n + 2k$
Why?
   $k$ changes to make diff. messages overlap
Recover?
   Reconstruct error polynomial, $E(X)$, and $P(x)$!
      Nonlinear equations.
Reconstruct $E(x)$ and $Q(x) = E(x)P(x)$. Linear Equations.
   Polynomial division! $P(x) = Q(x)/E(x)$!

Reed-Solomon codes. Welsh-Berlekamp Decoding. Perfection!
Cool.