Today: More Counting.

First Rule of counting: Objects from a sequence of choices:

n_i possibilitities for ith choice.

$n_1 \times n_2 \times \cdots \times n_k$ objects.

Second Rule of counting: If order does not matter.

Count with order.

Divide by number of orderings/sorted object.

Typically: $\binom{n}{k}$.

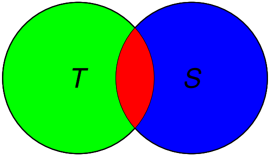
Inclusion/Exclusion: two sets of objects.

Add number of each and then subtract intersection of sets.

Sum Rule: If disjoint just add.

Simple Inclusion/Exclusion

Inclusion/Exclusion Rule:
For any S and T, $|S \cup T| = |S| + |T| - |S \cap T|$.



$S \cap T$

In T. $\Longrightarrow$ $|T|$

In S. $\Longrightarrow$ $+$ $|S|$

Elements in $S \cap T$ are counted twice.

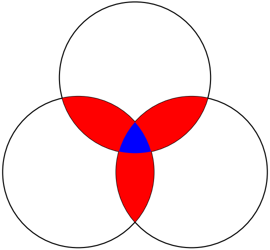
Subtract. $\Longrightarrow$ $-|S \cap T|$

$$|S \cup T| = |S| + |T| - |S \cap T|$$

Three way inclusion/exclusion.

A, B, C.

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$



In general.

Sets $A_1, \ldots, A_n$

$|\cup_i A_i| = \sum_i |A_i| - \sum_{i_1,i_2} |A_{i_1} \cup A_{j_1}| + \cdots + (-1)^n \sum_{i_1,\ldots i_n} |A_{i_1} \cup \cdots A_{i_n}|$

Element $x \in A_1 \cup ...A_k$. Maximal set that contains x!

How often is x counted?

$$k - \binom{k}{2} + \binom{k}{3} - \binom{k}{4} \cdots (-1)^{k+1} \binom{k}{k}.$$

That's 1? Yes.

$0 = (1-1)^k = 1 + \binom{k}{1}(-1)^1 + \cdots + \binom{k}{k}(-1)^k$

$\implies 1 = \binom{k}{1} - \binom{k}{2} + \cdots + \binom{k}{k}(-1)^{k+1}.$

Derangements

Permutations of $1,\ldots,n$? $n!$

Permutations where no item is in its proper place or fixed point.
Derangement or not?
123? No. All of 1, 2, 3 are fixed points.
213? No. 3 is a fixed point.
312? Yes.
231? Yes.
Any more? No. 132, 321 are not. 3! = 6 total.

How many? 2.

Count elements with at least one fixed?
$A_i =$ "permutations where i is fixed point."
$A_i \cap A_j =$ "permutations where i and j are fixed points."

$\binom{3}{1}(2!) - \binom{3}{2}(1) + \binom{3}{3}1.$

4

$3! - 4 = 2.$

General Case: for fun!

For n items.

$\binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \cdots + (-1)^{n-1}\binom{n}{n}$

Number of derangements is
$n! - \binom{n}{1}n-1! - \binom{n}{2}(n-1)! + \cdots + (-1)^{n-1}\binom{n}{n}$

Term i is $\frac{n!}{(n-i)!i!(-1)^i} \times (n-i)! = \frac{n!}{i!}(-1)^i$.

That is, number of derangements $= n! \times \sum_{i=0}^{n} \frac{(-1)^i}{i!}$.

Note: The summation is Taylor's expansion of e^{-1} as $n \to \infty$.

Roughly $1/e$ of the permutations are derangements.

Don't worry, this is for fun, though we will see derangements again.

And get an analagous conclusion.

Sampling...

Sample k items out of n

Without replacement:

Order matters: $n \times n-1 \times n-2 \ldots \times n-k+1 = \frac{n!}{(n-k)!}$

Order does not matter:

Second Rule: divide by number of orders – "k!"

$\implies \frac{n!}{(n-k)!k!}$.

"n choose k"

With Replacement.

Order matters: $n \times n \times \ldots n = n^k$

Order does not matter: Second rule ???

Problem: depends on how many of each item we chose!

Different number of unordered elts map to each unordered elt.

Unordered elt: $1,2,3$ $\quad$ 3! ordered elts map to it.

Unordered elt: $1,2,2$ $\quad$ $\frac{3!}{2!}$ ordered elts map to it.

How do we deal with this mess??

Splitting up some money....

How many ways can Bob and Alice split 5 dollars?
For each of 5 dollars pick Bob or Alice(2^5), divide out order ???

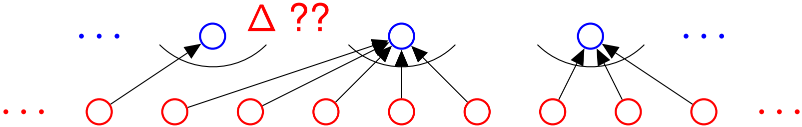
5 dollars for Bob and 0 for Alice:
one ordered set: (B, B, B, B, B).

4 for Bob and 1 for Alice:
5 ordered sets: (A, B, B, B, B) ; (B, A, B, B, B); ...

"Sorted" way to specify, first Alice's dollars, then Bob's.

(B, B, B, B, B): 1: (B,B,B,B,B)
(A, B, B, B, B): 5: (A,B,B,B,B),(B,A,B,B,B),(B,B,A,B,B),...
(A, A, B, B, B): $\binom{5}{2}$;(A,A,B,B,B),(A,B,A,B,B),(A,B,B,A,B),...

and so on.



Second rule of counting is no good here!

Splitting 5 dollars..

How many ways can Alice, Bob, and Eve split 5 dollars.

Alice gets 3, Bob gets 1, Eve gets 1: (A, A, A, B, E).

Separate Alice's dollars from Bob's and then Bob's from Eve's.

Five dollars are five stars: $\star\star\star\star\star$.

Alice: 2, Bob: 1, Eve: 2.
Stars and Bars: $\star\star|\star|\star\star$.

Alice: 0, Bob: 1, Eve: 4.
Stars and Bars: $|\star|\star\star\star\star$.

Each split "is" a sequence of stars and bars.
Each sequence of stars and bars "is" a split.

Counting Rule: if there is a one-to-one mapping between two sets they have the same size!

Stars and Bars.

How many different 5 star and 2 bar diagrams?

$| \star | \star \star \star \star$.

7 positions in which to place the 2 bars.

$\_ \_ \_ \_ \_ \_ \_$

Alice: 0; Bob 1; Eve: 4
$| \star | \star \star \star \star$.
Bars in first and third position.

Alice: 1; Bob 4; Eve: 0
$\star | \star \star \star \star |$.
Bars in second and seventh position.

$\binom{7}{2}$ ways to do so and
$\binom{7}{2}$ ways to split 5 dollars among 3 people.

Stars and Bars.

Ways to add up n numbers to sum to k? or

" k from n with replacement where order doesn't matter."

Correspondence: how many times is item i in the sample.

Correspondence: k stars $n-1$ bars gives assignment to n numbers!

$$\star\star|\star|\cdots|\star\star.$$

$n+k-1$ positions from which to choose $n-1$ bar positions.

$$\binom{n+k-1}{n-1}$$

Or: k unordered choices from set of n possibilities with replacement.
Sample with replacement where order doesn't matter.

Summary.

First rule: $n_1 \times n_2 \cdots \times n_3$.

k Samples with replacement from n items: n^k.
Sample without replacement: $\frac{n!}{(n-k)!}$

Second rule: when order doesn't matter (sometimes) can divide...

Sample without replacement and order doesn't matter: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.
"n choose k"

One-to-one rule: equal in number if one-to-one correspondence.
pause Bijection!

Sample k times from n objects with replacement and order doesn't matter: $\binom{k+n-1}{n-1}$.

Balls in bins.

"k Balls in n bins" $\equiv$ "k samples from n possibilities."

"indistinguishable balls" $\equiv$ "order doesn't matter"

"only one ball in each bin" $\equiv$ "without replacement"

5 balls into 10 bins
5 samples from 10 possibilities with replacement
Example: 5 digit numbers.

5 indistinguishable balls into 52 bins only one ball in each bin
5 samples from 52 possibilities without replacement
Example: Poker hands.

5 indistinguishable balls into 3 bins
5 samples from 3 possibilities with replacement and no order
Dividing 5 dollars among Alice, Bob and Eve.

Combinatorial Proofs.

Theorem: $\binom{n}{k} = \binom{n}{n-k}$

Proof: How many subsets of size k? $\binom{n}{k}$

How many subsets of size k?
Choose a subset of size $n-k$
and what's left out is a subset of size k.
Choosing a subset of size k is same
as choosing $n-k$ elements to not take.
$\implies \binom{n}{n-k}$ subsets of size k.

$\square$

Pascal's Triangle

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.

Foil (4 terms) on steroids:
  2^n terms: choose 1 or x from each factor of $(1+x)$.

Simplify: collect all terms corresponding to x^k.
  Coefficient of x^k is $\binom{n}{k}$: choose k factors where x is in product.

$$
\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3}
\end{array}
$$

Pascal's rule $\implies \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.
Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

How many size k subsets of $n+1$?
How many contain the first element?
Chose first element,
need to choose $k-1$ more from remaining n elements.
$\implies \binom{n}{k-1}$

How many don't contain the first element ?
Need to choose k elements from remaining n elts.
$\implies \binom{n}{k}$

So, $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$. $\square$

Combinatorial Proof.

Theorem: $\binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{k-1}{k-1}$.

Proof: Right hand side: Consider size k subset where i is the first element chosen.

$$\{1,\ldots,i,\underline{\ldots,n}\}$$

Must choose $k-1$ elements from $n-i$ remaining elements.
$\implies \binom{n-i}{k-1}$ such subsets.

Add them up to get the total number of subsets of size k.

This number is also $\binom{n}{k}$. The Left hand side.

$\square$

Binomial Theorem: $x = 1$

Theorem: $2^n = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{0}$

Proof: How many subsets of $\{1, \ldots, n\}$?
Construct a subset with sequence of n choices:
element i **is in** or **is not** in the subset: 2 poss.
First rule of counting: $2 \times 2 \cdots \times 2 = 2^n$ subsets.

How many subsets of $\{1, \ldots, n\}$?
$\binom{n}{i}$ ways to choose i elts of $\{1, \ldots, n\}$.
Sum over i to get total number of subsets..which is also 2^n. $\square$

Another Proof of Binomial Theorem.

Theorem A: $(x+y)^n = x^n\binom{n}{0} + x^{n-1}y^1\binom{n}{1} + \cdots + \binom{n}{n}y^n$.

Proof: True for $(x+y)^1$.
$(x+y)^n = x(x+y)^{n-1} + y(x+y)^{n-1}$.

Induction Hypothesis: $(x+y)^{n-1}$ has
$\binom{n-1}{k}$ terms of the form $x^{n-1-k}y^k$. First term gives $x^{n-k}y^k$.
$\binom{n-1}{k-1}$ terms of the form $x^{n-1-(k-1)}y^{k-1} = x^{n-k}y^{k-1}$
Second term gives $x^{n-k}y^k$.
$\implies \binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$
terms of the form $x^{n-k}y^k$ in $(x+y)^n$. $\square$

Theorem: $2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$.

Plugin $x = 1, y = 1$ or $(1+1)^n$ into Theorem A.

Ideas.

Inclusion/Exclusion:
  Union bound is too much, so subtract those in two sets.
  Those in three sets were added thrice, and then subtracted thrice.
    So add those in three sets.
    And so on.

k samples from n items with replacement without order.
  Map to k stars and $n-1$ bars to give count of each item.
  Count star/bar diagrams.

Combinatorial proof:
  Count the same thing in two ways to get valid equality.
    Sum of degrees is twice the number of edges.
    Each subset either has or doesn't have an element: 2^n
      Each subset has a size i : $\sum_i \binom{n}{i}$.