CS70: Lecture 20.

Expectation; Distributions; Independence

1. Expectation (Cont'd)
2. Important Distributions
3. Independence

Calculating $E[g(X)]$

Let $Y = g(X)$. Assume that we know the distribution of X.

We want to calculate $E[Y]$.

Method 1: We calculate the distribution of Y:

$$Pr[Y = y] = Pr[X \in g^{-1}(y)] \text{ where } g^{-1}(y) = \{x \in \Re : g(x) = y\}.$$

This is typically rather tedious!

Method 2: We use the following result.

Theorem:

$$E[g(X)] = \sum_x g(x) Pr[X = x].$$

Proof:

$$\begin{aligned} E[g(X)] &= \sum_\omega g(X(\omega)) Pr[\omega] = \sum_x \sum_{\omega \in X^{-1}(x)} g(X(\omega)) Pr[\omega] \\ &= \sum_x \sum_{\omega \in X^{-1}(x)} g(x) Pr[\omega] = \sum_x g(x) \sum_{\omega \in X^{-1}(x)} Pr[\omega] \\ &= \sum_x g(x) Pr[X = x]. \end{aligned}$$

□

An Example

Let X be uniform in $\{-2, -1, 0, 1, 2, 3\}$.

Let also $g(X) = X^2$. Then (method 2)

$$\begin{aligned} E[g(X)] &= \sum_{x=-2}^{3} x^2 \frac{1}{6} \\ &= \{4+1+0+1+4+9\}\frac{1}{6} = \frac{19}{6}. \end{aligned}$$

Method 1 - We find the distribution of $Y = X^2$:

$$Y = \begin{cases} 4, & \text{w.p. } \frac{2}{6} \\ 1, & \text{w.p. } \frac{2}{6} \\ 0, & \text{w.p. } \frac{1}{6} \\ 9, & \text{w.p. } \frac{1}{6}. \end{cases}$$

Thus,

$$E[Y] = 4\frac{2}{6} + 1\frac{2}{6} + 0\frac{1}{6} + 9\frac{1}{6} = \frac{19}{6}.$$

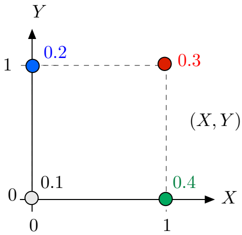
Calculating $E[g(X, Y, Z)]$

We have seen that $E[g(X)] = \sum_x g(x) Pr[X = x]$.

Using a similar derivation, one can show that

$$E[g(X,Y,Z)] = \sum_{x,y,z} g(x,y,z) Pr[X = x, Y = y, Z = z].$$

An Example. Let X, Y be as shown below:



$$(X,Y) = \begin{cases} (0,0), & \text{w.p. } 0.1 \\ (1,0), & \text{w.p. } 0.4 \\ (0,1), & \text{w.p. } 0.2 \\ (1,1), & \text{w.p. } 0.3 \end{cases}$$

$$\begin{aligned} E[\cos(2\pi X + \pi Y)] &= 0.1\cos(0) + 0.4\cos(2\pi) + 0.2\cos(\pi) + 0.3\cos(3\pi) \\ &= 0.1 \times 1 + 0.4 \times 1 + 0.2 \times (-1) + 0.3 \times (-1) = 0. \end{aligned}$$

Best Guess: Least Squares

If you only know the distribution of X, it seems that $E[X]$ is a 'good guess' for X.

The following result makes that idea precise.

Theorem
The value of a that minimizes $E[(X-a)^2]$ is $a = E[X]$.

Proof:

$$\begin{aligned}
E[(X-a)^2] &= E[(X-E[X]+E[X]-a)^2] \\
&= E[(X-E[X])^2+2(X-E[X])(E[X]-a)+(E[X]-a)^2] \\
&= E[(X-E[X])^2]+2(E[X]-a)E[X-E[X]]+(E[X]-a)^2 \\
&= E[(X-E[X])^2]+0+(E[X]-a)^2 \\
&\geq E[(X-E[X])^2].
\end{aligned}$$

□

Best Guess: Least Absolute Deviation

Thus $E[X]$ minimizes $E[(X-a)^2]$. It must be noted that the measure of the 'quality of the approximation' matters. The following result illustrates that point.

Theorem

The value of a that minimizes $E[|X-a|]$ is the *median* of X.

The median v of X is any real number such that

$$Pr[X \leq v] = Pr[X \geq v]$$

.

Proof:

$g(a) := E[|X-a|] = \sum_{x \leq a}(a-x)Pr[X=x] + \sum_{x \geq a}(x-a)Pr[X=x]$.

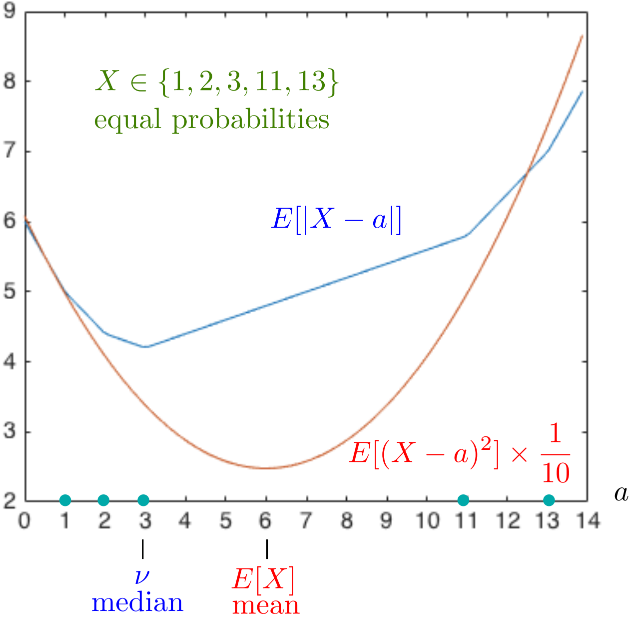
Thus, if $0 < \varepsilon << 1$,

$g(a+\varepsilon) = g(a) + \varepsilon Pr[X \leq a] - \varepsilon Pr[X \geq a]$.

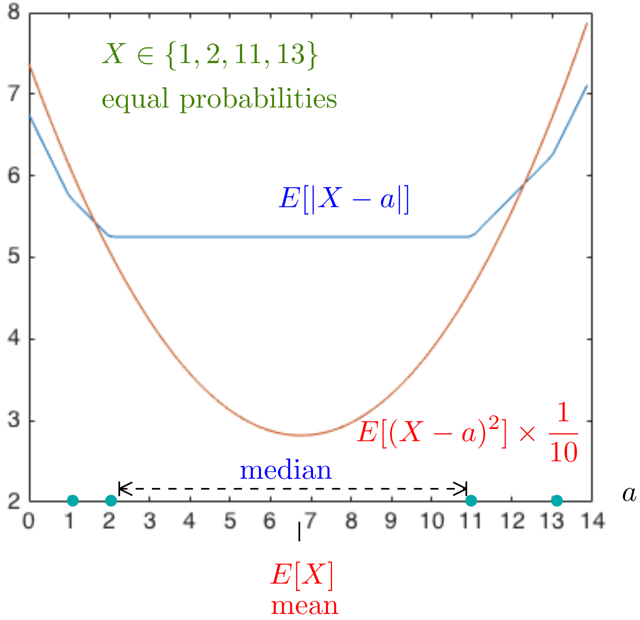
Hence, changing a cannot reduce $g(a)$ only if $Pr[X \leq a] = Pr[X \geq a]$.

□

Best Guess: Illustration

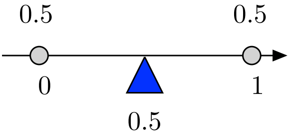


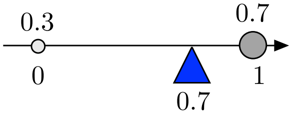
Best Guess: Another Illustration

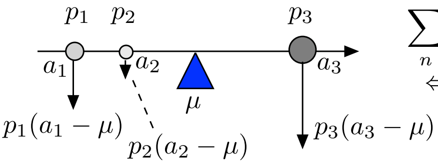


Center of Mass

The expected value has a *center of mass* interpretation:

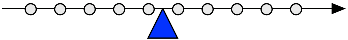


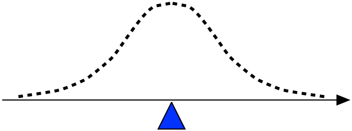




$$\sum_n p_n(a_n - \mu) = 0$$

$$\Leftrightarrow \mu = \sum_n a_n p_n = E[X]$$





Monotonicity

Definition
Let X, Y be two random variables on Ω. We write $X \leq Y$ if $X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$, and similarly for $X \geq Y$ and $X \geq a$ for some constant a.

Facts
(a) If $X \geq 0$, then $E[X] \geq 0$.
(b) If $X \leq Y$, then $E[X] \leq E[Y]$.

Proof
(a) If $X \geq 0$, every value a of X is nonnegative. Hence,

$$E[X] = \sum_a aPr[X = a] \geq 0.$$

(b) $X \leq Y \Rightarrow Y - X \geq 0 \Rightarrow E[Y] - E[X] = E[Y - X] \geq 0$.

□

Example:

$$B = \cup_m A_m \Rightarrow 1_B(\omega) \leq \sum_m 1_{A_m}(\omega) \Rightarrow Pr[\cup_m A_m] \leq \sum_m Pr[A_m].$$

Uniform Distribution

Roll a six-sided balanced die. Let X be the number of pips (dots). Then X is equally likely to take any of the values $\{1,2,\ldots,6\}$. We say that X is *uniformly distributed* in $\{1,2,\ldots,6\}$.

More generally, we say that X is uniformly distributed in $\{1,2,\ldots,n\}$ if $Pr[X=m]=1/n$ for $m=1,2,\ldots,n$.
In that case,

$$E[X]=\sum_{m=1}^{n} mPr[X=m]=\sum_{m=1}^{n} m\times\frac{1}{n}=\frac{1}{n}\frac{n(n+1)}{2}=\frac{n+1}{2}.$$

Geometric Distribution

Let's flip a coin with $Pr[H] = p$ until we get H.

For instance:

$$\omega_1 = H, \text{ or}$$
$$\omega_2 = T\ H, \text{ or}$$
$$\omega_3 = T\ T\ H, \text{ or}$$
$$\omega_n = T\ T\ T\ T\ \cdots\ T\ H.$$

Note that $\Omega = \{\omega_n, n = 1, 2, \ldots\}$.

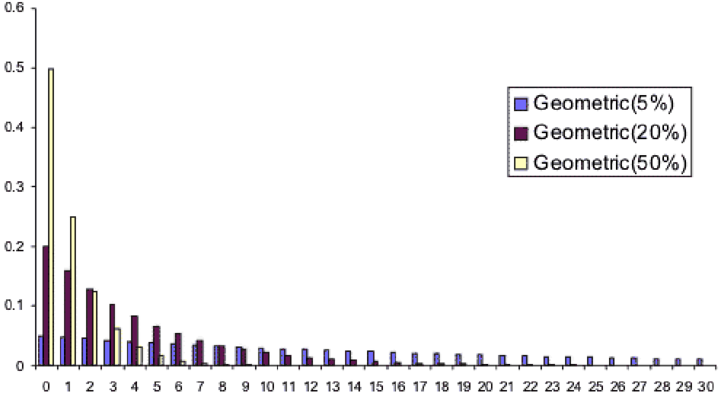
Let X be the number of flips until the first H. Then, $X(\omega_n) = n$.

Also,

$$Pr[X = n] = (1-p)^{n-1}p,\ n \geq 1.$$

Geometric Distribution

$$Pr[X = n] = (1 - p)^{n-1} p, n \geq 1.$$



Geometric Distribution

$$Pr[X = n] = (1 - p)^{n-1} p, n \geq 1.$$

Note that

$$\sum_{n=1}^{\infty} Pr[X_n] = \sum_{n=1}^{\infty} (1-p)^{n-1} p = p \sum_{n=1}^{\infty} (1-p)^{n-1} = p \sum_{n=0}^{\infty} (1-p)^n.$$

Now, if $|a| < 1$, then $S := \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$. Indeed,

$$\begin{aligned} S &= 1 + a + a^2 + a^3 + \cdots \\ aS &= a + a^2 + a^3 + a^4 + \cdots \\ (1-a)S &= 1 + a - a + a^2 - a^2 + \cdots = 1. \end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} Pr[X_n] = p \, \frac{1}{1-(1-p)} = 1.$$

Geometric Distribution: Expectation

$$X =_D G(p), \text{ i.e., } Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$

One has

$$E[X] = \sum_{n=1}^{\infty} nPr[X = n] = \sum_{n=1}^{\infty} n(1-p)^{n-1}p.$$

Thus,

$$\begin{aligned}
E[X] &= p + 2(1-p)p + 3(1-p)^2p + 4(1-p)^3p + \cdots \\
(1-p)E[X] &= (1-p)p + 2(1-p)^2p + 3(1-p)^3p + \cdots \\
pE[X] &= p + (1-p)p + (1-p)^2p + (1-p)^3p + \cdots \\
& \qquad \text{by subtracting the previous two identities} \\
&= \sum_{n=1}^{\infty} Pr[X = n] = 1.
\end{aligned}$$

Hence,

$$E[X] = \frac{1}{p}.$$

Geometric Distribution: Memoryless

Let X be $G(p)$. Then, for $n \geq 0$,

$$Pr[X > n] = Pr[\text{ first } n \text{ flips are } T] = (1-p)^n.$$

Theorem

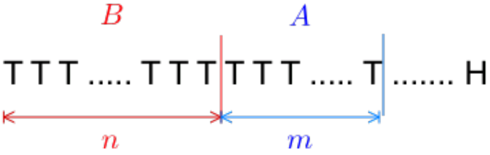
$$Pr[X > n+m | X > n] = Pr[X > m], m, n \geq 0.$$

Proof:

$$\begin{aligned} Pr[X > n+m | X > n] &= \frac{Pr[X > n+m \text{ and } X > n]}{Pr[X > n]} \\ &= \frac{Pr[X > n+m]}{Pr[X > n]} \\ &= \frac{(1-p)^{n+m}}{(1-p)^n} = (1-p)^m \\ &= Pr[X > m]. \end{aligned}$$

Geometric Distribution: Memoryless - Interpretation

$$Pr[X > n+m | X > n] = Pr[X > m], m, n \geq 0.$$



$$Pr[X > n+m | X > n] = Pr[A|B] = Pr[A] = Pr[X > m].$$

The coin is memoryless, therefore, so is X.

Geometric Distribution: Yet another look

Theorem: For a r.v. X that takes the values $\{0,1,2,\ldots\}$, one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$

[See later for a proof.]

If $X = G(p)$, then $Pr[X \geq i] = Pr[X > i-1] = (1-p)^{i-1}$.

Hence,

$$E[X] = \sum_{i=1}^{\infty} (1-p)^{i-1} = \sum_{i=0}^{\infty} (1-p)^{i} = \frac{1}{1-(1-p)} = \frac{1}{p}.$$

Expected Value of Integer RV

Theorem: For a r.v. X that takes values in $\{0, 1, 2, \ldots\}$, one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$

Proof: One has

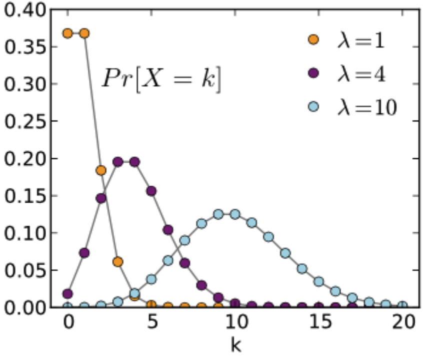
$$\begin{aligned}
E[X] &= \sum_{i=1}^{\infty} i \times Pr[X = i] \\
&= \sum_{i=1}^{\infty} i\{Pr[X \geq i] - Pr[X \geq i+1]\} \\
&= \sum_{i=1}^{\infty} \{i \times Pr[X \geq i] - i \times Pr[X \geq i+1]\} \\
&= \sum_{i=1}^{\infty} \{i \times Pr[X \geq i] - (i-1) \times Pr[X \geq i]\} \\
&= \sum_{i=1}^{\infty} Pr[X \geq i].
\end{aligned}$$

□

Poisson

Experiment: flip a coin n times. The coin is such that $Pr[H] = \lambda/n$.
Random Variable: X - number of heads. Thus, $X = B(n, \lambda/n)$.

Poisson Distribution is distribution of X "for large n."



Poisson

Experiment: flip a coin n times. The coin is such that $Pr[H] = \lambda/n$.
Random Variable: X - number of heads. Thus, $X = B(n, \lambda/n)$.
Poisson Distribution is distribution of X "for large n."
We expect $X \ll n$. For $m \ll n$ one has

$$\begin{aligned}
Pr[X = m] &= \binom{n}{m} p^m (1-p)^{n-m}, \text{ with } p = \lambda/n \\
&= \frac{n(n-1)\cdots(n-m+1)}{m!} \left(\frac{\lambda}{n}\right)^m \left(1 - \frac{\lambda}{n}\right)^{n-m} \\
&= \frac{n(n-1)\cdots(n-m+1)}{n^m} \frac{\lambda^m}{m!} \left(1 - \frac{\lambda}{n}\right)^{n-m} \\
&\approx^{(1)} \frac{\lambda^m}{m!} \left(1 - \frac{\lambda}{n}\right)^{n-m} \approx^{(2)} \frac{\lambda^m}{m!} \left(1 - \frac{\lambda}{n}\right)^{n} \approx \frac{\lambda^m}{m!} e^{-\lambda}.
\end{aligned}$$

For (1) we used $m \ll n$; for (2) we used $(1 - a/n)^n \approx e^{-a}$.

Poisson Distribution: Definition and Mean

Definition Poisson Distribution with parameter $\lambda > 0$

$$X = P(\lambda) \Leftrightarrow Pr[X = m] = \frac{\lambda^m}{m!} e^{-\lambda}, m \geq 0.$$

Fact: $E[X] = \lambda$.

Proof:

$$\begin{aligned} E[X] &= \sum_{m=1}^{\infty} m \times \frac{\lambda^m}{m!} e^{-\lambda} = e^{-\lambda} \sum_{m=1}^{\infty} \frac{\lambda^m}{(m-1)!} \\ &= e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^{m+1}}{m!} = e^{-\lambda} \lambda \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \\ &= e^{-\lambda} \lambda e^{\lambda} = \lambda. \end{aligned}$$

$\square$

Simeon Poisson

The Poisson distribution is named after:

Siméon Poisson

Siméon Denis Poisson (1781–1840)

Independent Random Variables.

Definition: Independence

The random variables X and Y are **independent** if and only if

$$Pr[Y = b | X = a] = Pr[Y = b], \text{ for all } a \text{ and } b.$$

Fact:

X, Y are independent if and only if

$$Pr[X = a, Y = b] = Pr[X = a]Pr[Y = b], \text{ for all } a \text{ and } b.$$

Obvious.

Independence: Examples

Example 1
Roll two die. X, Y = number of pips on the two dice. X, Y are independent.

Indeed: $Pr[X = a, Y = b] = \frac{1}{36}, Pr[X = a] = Pr[Y = b] = \frac{1}{6}$.

Example 2
Roll two die. X = total number of pips, Y = number of pips on die 1 minus number on die 2. X and Y are not independent.

Indeed: $Pr[X = 12, Y = 1] = 0 \neq Pr[X = 12]Pr[Y = 1] > 0$.

Example 3
Flip a fair coin five times, X = number of Hs in first three flips, Y = number of Hs in last two flips. X and Y are independent.

Indeed:

$$Pr[X = a, Y = b] = \binom{3}{a}\binom{2}{b}2^{-5} = \binom{3}{a}2^{-3} \times \binom{2}{b}2^{-2} = Pr[X = a]Pr[Y = b].$$

A useful observation about independence

Theorem

X and Y are independent if and only if

$$Pr[X \in A, Y \in B] = Pr[X \in A]Pr[Y \in B] \text{ for all } A, B \subset \Re.$$

Proof:

If ($\Leftarrow$): Choose $A = \{a\}$ and $B = \{b\}$.

This shows that $Pr[X = a, Y = b] = Pr[X = a]Pr[Y = b]$.

Only if ($\Rightarrow$):

$$\begin{aligned}
&Pr[X \in A, Y \in B] \\
&= \sum_{a \in A} \sum_{b \in B} Pr[X = a, Y = b] = \sum_{a \in A} \sum_{b \in B} Pr[X = a]Pr[Y = b] \\
&= \sum_{a \in A} [\sum_{b \in B} Pr[X = a]Pr[Y = b]] = \sum_{a \in A} Pr[X = a][\sum_{b \in B} Pr[Y = b]] \\
&= \sum_{a \in A} Pr[X = a]Pr[Y \in B] = Pr[X \in A]Pr[Y \in B].
\end{aligned}$$

$\square$

Functions of Independent random Variables

Theorem Functions of independent RVs are independent
Let X, Y be independent RV. Then

$$f(X) \text{ and } g(Y) \text{ are independent, for all } f(\cdot), g(\cdot).$$

Proof:
Recall the definition of inverse image:

$$h(z) \in C \Leftrightarrow z \in h^{-1}(C) := \{z \mid h(z) \in C\}. \tag{1}$$

Now,

$$\begin{aligned}
&Pr[f(X) \in A, g(Y) \in B] \\
&\quad = Pr[X \in f^{-1}(A), Y \in g^{-1}(B)], \text{ by } (1) \\
&\quad = Pr[X \in f^{-1}(A)]Pr[Y \in g^{-1}(B)], \text{ since } X, Y \text{ ind.} \\
&\quad = Pr[f(X) \in A]Pr[g(Y) \in B], \text{ by } (1).
\end{aligned}$$

$\square$

Mean of product of independent RV

Theorem

Let X, Y be independent RVs. Then

$$E[XY] = E[X]E[Y].$$

Proof:

Recall that $E[g(X,Y)] = \sum_{x,y} g(x,y)Pr[X = x, Y = y]$. Hence,

$$\begin{aligned}
E[XY] &= \sum_{x,y} xyPr[X = x, Y = y] = \sum_{x,y} xyPr[X = x]Pr[Y = y], \text{ by ind.} \\
&= \sum_{x}[\sum_{y} xyPr[X = x]Pr[Y = y]] = \sum_{x}[xPr[X = x](\sum_{y} yPr[Y = y])] \\
&= \sum_{x}[xPr[X = x]E[Y]] = E[X]E[Y].
\end{aligned}$$

□

Examples

(1) Assume that X, Y, Z are (pairwise) independent, with $E[X] = E[Y] = E[Z] = 0$ and $E[X^2] = E[Y^2] = E[Z^2] = 1$.

Then

$$\begin{aligned} E[(X+2Y+3Z)^2] &= E[X^2+4Y^2+9Z^2+4XY+12YZ+6XZ] \\ &= 1+4+9+4\times 0+12\times 0+6\times 0 \\ &= 14. \end{aligned}$$

(2) Let X, Y be independent and $U[1,2,\ldots n]$. Then

$$\begin{aligned} E[(X-Y)^2] &= E[X^2+Y^2-2XY] = 2E[X^2]-2E[X]^2 \\ &= \frac{1+3n+2n^2}{3} - \frac{(n+1)^2}{2}. \end{aligned}$$

Mutually Independent Random Variables

Definition

X, Y, Z are mutually independent if

$$Pr[X = x, Y = y, Z = z] = Pr[X = x]Pr[Y = y]Pr[Z = z], \text{ for all } x, y, z.$$

Theorem
The events $A, B, C, \ldots$ are pairwise (resp. mutually) independent iff the random variables $1_A, 1_B, 1_C, \ldots$ are pairwise (resp. mutually) independent.
Proof:

$$Pr[1_A = 1, 1_B = 1, 1_C = 1] = Pr[A \cap B \cap C], \ldots$$

□

Functions of pairwise independent RVs

If X, Y, Z are pairwise independent, but not mutually independent, it may be that

$$f(X) \text{ and } g(Y, Z) \text{ are not independent.}$$

Example 1: Flip two fair coins, $X = 1\{\text{coin 1 is } H\}, Y = 1\{\text{coin 2 is } H\}, Z = X \oplus Y$. Then, X, Y, Z are pairwise independent. Let $g(Y, Z) = Y \oplus Z$. Then $g(Y, Z) = X$ is not independent of X.

Example 2: Let A, B, C be pairwise but not mutually independent in a way that A and $B \cap C$ are not independent. Let $X = 1_A, Y = 1_B, Z = 1_C$. Choose $f(X) = X, g(Y, Z) = YZ$.

Functions of mutually independent RVs

One has the following result:

Theorem

Functions of disjoint collections of mutually independent random variables are mutually independent.

Example:

Let $\{X_n, n \geq 1\}$ be mutually independent. Then,

$Y_1 := X_1 X_2 (X_3 + X_4)^2, Y_2 := \max\{X_5, X_6\} - \min\{X_7, X_8\}, Y_3 := X_9 \cos(X_{10} + X_{11})$ are mutually independent.

Proof:

Let $B_1 := \{(x_1, x_2, x_3, x_4) \mid x_1 x_2 (x_3 + x_4)^2 \in A_1\}$. Similarly for B_2, B_3. Then

$$
\begin{aligned}
& Pr[Y_1 \in A_1, Y_2 \in A_2, Y_3 \in A_3] \\
& \quad = Pr[(X_1, \ldots, X_4) \in B_1, (X_5, \ldots, X_8) \in B_2, (X_9, \ldots, X_{11}) \in B_3] \\
& \quad = Pr[(X_1, \ldots, X_4) \in B_1] Pr[(X_5, \ldots, X_8) \in B_2] Pr[(X_9, \ldots, X_{11}) \in B_3] \\
& \quad = Pr[Y_1 \in A_1] Pr[Y_2 \in A_2] Pr[Y_3 \in A_3]
\end{aligned}
$$

□

Operations on Mutually Independent Events

Theorem

Operations on disjoint collections of mutually independent events produce mutually independent events.

For instance, if A, B, C, D, E are mutually independent, then $A \Delta B, C \setminus D, \bar{E}$ are mutually independent.

Proof:

$1_{A \Delta B} = f(1_A, 1_B)$ where
$f(0,0) = 0, f(1,0) = 1, f(0,1) = 1, f(1,1) = 0$

$1_{C \setminus D} = g(1_C, 1_D)$ where
$g(0,0) = 0, g(1,0) = 1, g(0,1) = 0, g(1,1) = 0$

$1_{\bar{E}} = h(1_E)$ where
$h(0) = 1$ and $h(1) = 0$.

Hence, $1_{A \Delta B}, 1_{C \setminus D}, 1_{\bar{E}}$ are functions of mutually independent RVs. Thus, those RVs are mutually independent. Consequently, the events of which they are indicators are mutually independent. □

Product of mutually independent RVs

Theorem

Let $X_1, \ldots, X_n$ be mutually independent RVs. Then,

$$E[X_1 X_2 \cdots X_n] = E[X_1]E[X_2] \cdots E[X_n].$$

Proof:

Assume that the result is true for n. (It is true for $n = 2$.)

Then, with $Y = X_1 \cdots X_n$, one has

$$\begin{aligned} E[X_1 \cdots X_n X_{n+1}] &= E[Y X_{n+1}], \\ &= E[Y]E[X_{n+1}], \\ &\quad \text{because } Y, X_{n+1} \text{ are independent} \\ &= E[X_1] \cdots E[X_n]E[X_{n+1}]. \end{aligned}$$

□

Summary.

Expectation; Distributions; Independence

Expectation:

- $E[X] := \sum_a a Pr[X = a]$.
- Expectation is Linear.

Distributions:

- $G(p) : E[X] = 1/p$;
- $B(n,p) : E[X] = np$;
- $P(\lambda) : E[X] = \lambda$

Independence:

- X, Y independent
  $\Leftrightarrow Pr[X \in A, Y \in B] = Pr[X \in A] Pr[Y \in B]$
- Then, $f(X), g(Y)$ are independent
  and $E[XY] = E[X]E[Y]$
- Mutual independence ....