CS70: Lecture 27.

Probability Review

1. True or False
2. Matching
3. Questions
4. Common Mistakes

True or False (1)

1. Ω and A are independent.
2. $Pr[A \cap B] = Pr[A] + Pr[B] - Pr[A \cup B]$.
3. $Pr[A \setminus B] \geq Pr[A] - Pr[B]$.
4. $Pr[A|B] = Pr[B|A]$ implies $Pr[A] = Pr[B]$.

True or False (1) - Answers

1. Ω and A are independent. True
2. $Pr[A \cap B] = Pr[A] + Pr[B] - Pr[A \cup B]$. True
3. $Pr[A \setminus B] \geq Pr[A] - Pr[B]$. True
4. $Pr[A|B] = Pr[B|A]$ implies $Pr[A] = Pr[B]$. False: $Pr[A \cap B]$ can be 0

True or False (2)

1. $X_1, \dots, X_n$ i.i.d. $\implies var(\frac{X_1 + \cdots + X_n}{n}) = var(X_1)$.
2. $Pr[|X - a| \geq b] \leq \frac{E[(X-a)^2]}{b^2}$.
3. $X_1, \dots, X_n$ i.i.d. $\implies \frac{X_1 + \cdots + X_n - nE[X_1]}{n\sigma(X_1)} \to \mathcal{N}(0,1)$.
4. $X = Expo(\lambda) \implies Pr[X > 5 | X > 3] = Pr[X > 2]$.

True or False (2) - Answers

1. $X_1, \dots, X_n$ i.i.d. $\implies var(\frac{X_1 + \cdots + X_n}{n}) = var(X_1)$. False: $\times \frac{1}{n}$
2. $Pr[|X - a| \geq b] \leq \frac{E[(X-a)^2]}{b^2}$. True
3. $X_1, \dots, X_n$ i.i.d. $\implies \frac{X_1 + \cdots + X_n - nE[X_1]}{n\sigma(X_1)} \to \mathcal{N}(0,1)$. False: $\sqrt{n}$
4. $X = Expo(\lambda) \implies Pr[X > 5 | X > 3] = Pr[X > 2]$. True: $\frac{\exp\{-\lambda 5\}}{\exp\{-\lambda 3\}} = \exp\{-\lambda 2\}$.

True or False (3)

When $n \gg 1$, one has

1. $[A_n - 2\sigma\frac{1}{n}, A_n + 2\sigma\frac{1}{n}] =$ 95%-CI for μ.
2. $[A_n - 2\sigma\frac{1}{\sqrt{n}}, A_n + 2\sigma\frac{1}{\sqrt{n}}] =$ 95%-CI for μ.
3. If $0.3 < \sigma < 3$, then
   $[A_n - 0.6\frac{1}{\sqrt{n}}, A_n + 0.6\frac{1}{\sqrt{n}}] =$ 95%-CI for μ.
4. If $0.3 < \sigma < 3$, then
   $[A_n - 6\frac{1}{\sqrt{n}}, A_n + 6\frac{1}{\sqrt{n}}] =$ 95%-CI for μ.

True or False (3) - Answers

When $n \gg 1$, one has

1. $[A_n - 2\sigma\frac{1}{n}, A_n + 2\sigma\frac{1}{n}] = 95\%$-CI for μ. False
2. $[A_n - 2\sigma\frac{1}{\sqrt{n}}, A_n + 2\sigma\frac{1}{\sqrt{n}}] = 95\%$-CI for μ. True
3. If $0.3 < \sigma < 3$, then $[A_n - 0.6\frac{1}{\sqrt{n}}, A_n + 0.6\frac{1}{\sqrt{n}}] = 95\%$-CI for μ. False
4. If $0.3 < \sigma < 3$, then $[A_n - 6\frac{1}{\sqrt{n}}, A_n + 6\frac{1}{\sqrt{n}}] = 95\%$-CI for μ. True

Match Items (1)

[1] $Pr[X \geq a] \leq \frac{E[f(X)]}{f(a)}$.

[2] $Pr[|X - E[X]| > a] \leq \frac{var[X]}{a^2}$

[3] $Pr[X \geq a] \leq \min_{\theta > 0} \frac{E[e^{\theta X}]}{e^{\theta a}}$

[4] $g(\cdot)$ convex $\Rightarrow E[g(X)] \geq g(E[X])$

[5] $E[Y] + \frac{cov(X,Y)}{var(X)}(X - E[X])$.

[6] $\sum_y yPr[Y = y|X = x]$

[7] $Pr[|\frac{X_1 + \cdots + X_n}{n} - E[X_1]| \geq \epsilon] \to 0$,

[8] $E[(Y - E[Y|X])h(X)] = 0$.

1. WLLN
2. MMSE
3. Projection property

Match Items (1) - Answers

[1] $Pr[X \geq a] \leq \frac{E[f(X)]}{f(a)}$.

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[8] $E[(Y - E[Y|X])h(X)] = 0$.

1. WLLN [7]
2. MMSE [6]
3. Projection property [8]

Match Items (2)

[1] $Pr[X \geq a] \leq \frac{E[f(X)]}{f(a)}$.

[2] $Pr[|X - E[X]| > a] \leq \frac{var[X]}{a^2}$

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[8] $E[(Y - E[Y|X])h(X)] = 0$.

1. Chebyshev
2. LLSE
3. Markov's inequality

Match Items (2) - Answers

[1] $Pr[X \geq a] \leq \frac{E[f(X)]}{f(a)}$.

[2] $Pr[|X - E[X]| > a] \leq \frac{var[X]}{a^2}$

[3] $Pr[X \geq a] \leq \min_{\theta > 0} \frac{E[e^{\theta X}]}{e^{\theta a}}$

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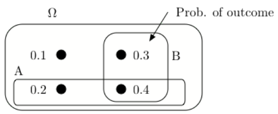
[7] $Pr[|\frac{X_1 + \cdots + X_n}{n} - E[X_1]| \geq \epsilon] \to 0$,

[8] $E[(Y - E[Y|X])h(X)] = 0$.

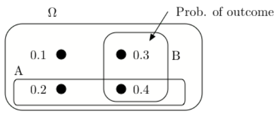
1. Chebyshev [2]
2. LLSE [5]
3. Markov's inequality [1]

Questions (1)



1. What is $P[A|B]$?
2. What is $Pr[B|A]$?
3. Are A and B positively correlated?

Questions (1) - Solutions



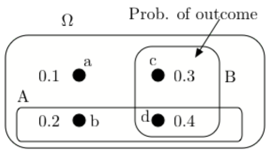
1. What is $P[A|B]$?

   $Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]} = \frac{0.4}{0.7}$
2. What is $Pr[B|A]$?

   $Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]} = \frac{0.4}{0.6}$
3. Are A and B positively correlated?

   No. $Pr[A \cap B] = 0.4 < Pr[A]Pr[B] = 0.6 \times 0.7$.

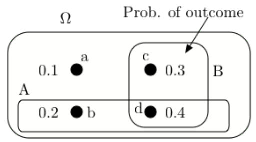
Questions (2)



ω	$X(\omega)$	$Y(\omega)$
a	0	0
b	1	0
c	0	2
d	1	2

4. What is $E[Y|X]$?
5. What is $cov(X,Y)$?
6. What is $L[Y|X]$?

Questions (2) - Solutions



ω	$X(\omega)$	$Y(\omega)$
a	0	0
b	1	0
c	0	2
d	1	2

4. What is $E[Y|X]$?

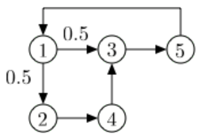
$$\begin{aligned} E[Y|X=0] &= 0 \times Pr[Y=0|X=0] + 2 \times Pr[Y=2|X=0] \\ &= 2 \times \frac{0.3}{0.4} = 1.5 \\ E[Y|X=1] &= 0 \times Pr[Y=0|X=1] + 2 \times Pr[Y=2|X=1] \\ &= 2 \times \frac{0.4}{0.6} = 1.33 \end{aligned}$$

5. What is $cov(X,Y)$?

   $cov(X,Y) = E[XY] - E[X]E[Y] = 0.8 - 0.6 \times 1.4 = -0.04$
6. What is $L[Y|X]$?

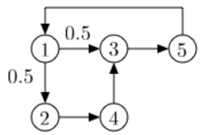
   $L[Y|X] = E[Y] + \frac{cov(X,Y)}{var(X)}(X - E[X]) = 1.4 + \frac{-0.04}{0.6 \times 0.4}(X - 0.6)$

Questions (3)



7. Is this Markov chains irreducible?
8. Is this Markov chain periodic?
9. Does π_n converge to a value independent of π_0?
10. Does $\frac{1}{n}\sum_{m=1}^{n-1} 1\{X_m = 3\}$ converge as $n \to \infty$?
11. Calculate π.

Questions (3) - Solutions

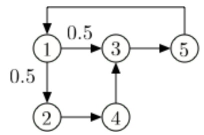


7. Is this Markov chains irreducible? Yes.
8. Is this Markov chain periodic?

   No. The return times to 3 are $\{3, 5, ..\}$: coprime!
9. Does π_n converge to a value independent of π_0? Yes!
10. Does $\frac{1}{n}\sum_{m=1}^{n-1} 1\{X_m = 3\}$ converge as $n \to \infty$? Yes!
11. Calculate π.

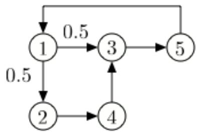
    Let $a = \pi(1)$. Then $a = \pi(5), \pi(2) = 0.5a, \pi(4) = \pi(2) = 0.5a, \pi(3) = 0.5\pi(1) + \pi(4) = a$. Thus, $\pi = [a, 0.5a, a, 0.5a, a] = [1, 0.5, 1, 0.5, 1]a$, so $a = 1/4$.

Questions (4)



12. Write the first step equations for calculating the mean time from 1 to 4.
13. Solve these equations.

Questions (4) - Solutions



12. Write the first step equations for calculating the mean time from 1 to 4.

$$\beta(1) = 1 + 0.5\beta(2) + 0.5\beta(3)$$
$$\beta(2) = 1$$
$$\beta(3) = 1 + \beta(5)$$
$$\beta(5) = 1 + \beta(1).$$

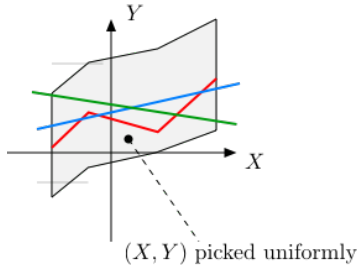
13. Solve these equations.

$$\beta(1) = 1 + 0.5 \times 1 + 0.5 \times (1 + (1 + \beta(1)))$$
$$= 2.5 + 0.5\beta(1).$$

Hence, $\beta(1) = 5$.

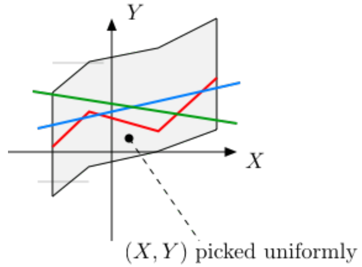
Questions (5)

14. Which is $L[Y|X]$? Blue, red or green?



Questions (5) - Solutions

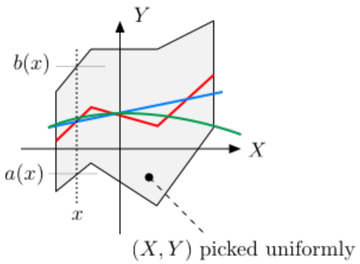
14. Which is $L[Y|X]$? Blue, red or green?



Answer: Blue.
Cannot be red (not a straight line).
Cannot be green: X and Y are clearly positively correlated.

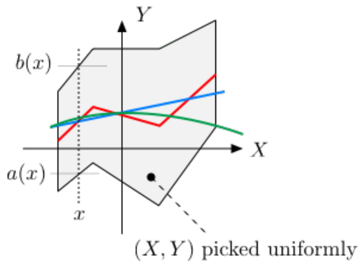
Questions (6)

15. Which is $E[Y|X]$? Blue, red or green?



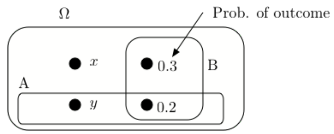
Questions (6) - Solutions

15. Which is $E[Y|X]$? Blue, red or green?



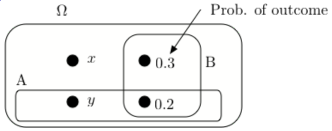
Answer: Red.
Given $X = x$, $Y = U[a(x), b(x)]$. Thus, $E[Y|X = x] = \frac{a(x)+b(x)}{2}$.

Questions (7)



16. Find (x, y) so that A and B are independent.
17. Find the value of x that maximizes $Pr[B|A]$.

Questions (7) - Solutions



16. Find (x, y) so that A and B are independent.
    We need
    $$Pr[A \cap B] = Pr[A]Pr[B]$$
    That is,
    $$0.2 = (y + 0.2) \times 0.5 = 0.5y + 0.1$$
    Hence,
    $$y = 0.2 \text{ and } x = 0.3.$$

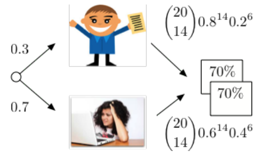
17. Find the value of x that maximizes $Pr[B|A]$.
    Obviously, it is $x = 0.5$.

Questions (8)

18. A CS70 student is great w.p. 0.3 and good w.p. 0.7. A great student solves each question correctly w.p. 0.8 whereas a good student does it w.p. 0.6. One student got right 70% of the 20 questions on midterm. What is the expected score of the student on the final?

Questions (8) - Solutions

18. A CS70 student is great w.p. 0.3 and good w.p. 0.7. A great student solves each question correctly w.p. 0.8 whereas a good student does it w.p. 0.6. One student got right 70% of the 20 questions on midterm. What is the expected score of the student on the final?



$$p := Pr[\text{great}|\text{score}] = \frac{0.3\binom{20}{14}0.8^{14}0.2^{6}}{0.3\binom{20}{14}0.8^{14}0.2^{6} + 0.7\binom{20}{14}0.6^{14}0.4^{6}}$$
$$= \frac{(0.3)0.8^{14}0.2^{6}}{(0.3)0.8^{14}0.2^{6} + (0.7)0.6^{14}0.4^{6}} \approx 0.27$$
$$\text{Expected score } = p80\% + (1-p)60\% \approx 65\%.$$

Questions (9)

19. You roll a balanced six-sided die 20 times. Use CLT to upper-bound the probability that the total number of dots exceeds 85.

Questions (9) - Solutions

19. You roll a balanced six-sided die 20 times. Use CLT to upper-bound the probability that the total number of dots exceeds 85.

Let $X = X_1 + \cdots + X_{20}$ be the total number of dots.
Then
$$\frac{X - 70}{\sigma\sqrt{20}} \approx \mathcal{N}(0,1)$$
where
$$\sigma^2 = var(X_1) = (1/6)\sum_{m=1}^{6} m^2 - (3.5)^2 \approx 2.9 = 1.7^2.$$
Now,
$$\begin{aligned} Pr[X > 85] &= Pr[X - 70 > 15] \\ &= Pr[\frac{X-70}{1.7 \times 4.5} > \frac{15}{1.7 \times 4.5}] \\ &= Pr[\frac{X-70}{1.7 \times 4.5} > 2] \approx 2.5\%. \end{aligned}$$

Questions (10)

20. You roll a balanced six-sided die 20 times. Use Chebyshev to upper-bound the probability that the total number of dots exceeds 85.

Questions (10) - Solutions

20. You roll a balanced six-sided die 20 times. Use Chebyshev to upper-bound the probability that the total number of dots exceeds 85.

Let $X = X_1 + \cdots + X_{20}$ be the total number of dots.
Then

$$\begin{aligned} Pr[X > 85] &= Pr[X - 70 > 15] \leq Pr[|X - 70| > 15] \\ &\leq \frac{var(X)}{15^2}. \end{aligned}$$

Now,

$$var(X) = 20 var(X_1) = 20 \times 2.9 = 58.$$

Hence,

$$Pr[X > 85] \leq \frac{58}{15^2} \approx 0.26.$$

Questions (11)

21. Let X, Y, Z be i.i.d. $Expo(1)$. Find $L[X|X + 2Y + 3Z]$.
22. Let X, Y, Z be i.i.d. $Expo(1)$. Calculate $E[X + Z|X + Y]$.
23. Let X, Y, Z be i.i.d. $Expo(1)$. Calculate $L[X + Z|X + Y]$.

Questions (11) - Solutions

21. Let X, Y, Z be i.i.d. $Expo(1)$. Find $L[X|X + 2Y + 3Z]$.
Let $V = X + 2Y + 3Z$. One finds

$$\begin{aligned} &L[X|V] = E[X] + \frac{cov(X, V)}{var(V)}(V - E[V]) \\ &E[X] = 1, E[V] = 6 \\ &cov(X, V) = var(X) = 1 \\ &var(V) = 1 + 4 + 9 = 14. \end{aligned}$$

Hence,

$$L[X|V] = 1 + \frac{1}{14}(V - 6).$$

22. Let X, Y, Z be i.i.d. $Expo(1)$. Calculate $E[X + Z|X + Y]$.

$$\begin{aligned} E[X + Z|X + Y] &= E[X|X + Y] + E[Z] \\ &= \frac{1}{2}(X + Y) + 1. \end{aligned}$$

23. Let X, Y, Z be i.i.d. $Expo(1)$. Calculate $L[X + Z|X + Y]$.

$$L[X + Z|X + Y] = \frac{1}{2}(X + Y) + 1.$$

Questions (12)

24. You roll a balanced die.

    You start with \$1.00.

    Every time you get a 6, your fortune is multiplied by 10.

    Every time you do not get a 6, your fortune is divided by 2.

    Let X_n be your fortune at the start of step n,

    Calculate $E[X_n]$.

Questions (12) - Solutions

24. You roll a balanced die.

    You start with \$1.00.

    Every time you get a 6, your fortune is multiplied by 10.

    Every time you do not get a 6, your fortune is divided by 2.

    Let X_n be your fortune at the start of step n,

    Calculate $E[X_n]$.

We have $X_1 = 1$. Also,

$$\begin{aligned} E[X_{n+1}|X_n] &= X_n \times [10\frac{1}{6} + 0.5 \times \frac{5}{6}] \\ &= \rho X_n, \rho = 10\frac{1}{6} + 0.5 \times \frac{5}{6} \approx 2.1. \end{aligned}$$

Hence,

$$E[X_{n+1}] = \rho E[X_n], n \geq 1.$$

Thus,

$$E[X_n] = \rho^{n-1}, n \geq 1.$$

Questions (13)

25. The lifespans of good lightbulbs are exponentially distributed with mean 1 year. Those of defective bulbs are exponentially distributed with mean 0.8. All the bulbs in one batch are equally likely to be good or defective. You test one bulb and note that it burns out after 0.6 year. (a) What is the probability you got a batch of good bulbs? (b) What is the expected lifespan of another bulb in that batch?

Questions (13) - Solutions

25. The lifespans of good lightbulbs are exponentially distributed with mean 1 year. Those of defective bulbs are exponentially distributed with mean 0.8. All the bulbs in one batch are equally likely to be good or defective. You test one bulb and note that it burns out after 0.6 year. (a) What is the probability you got a batch of good bulbs? (b) What is the expected lifespan of another bulb in that batch?

Hint: If $X = Expo(\lambda), f_X(x) = \lambda e^{-\lambda x} 1\{x > 0\}, E[X] = 1/\lambda$.

Let X be the lifespan of a bulb, G the event that it is good, and B the event that it is bad.

$$(a) \quad p := Pr[G | X \in (0.6, 0.6+\delta)] = \frac{0.5 Pr[X \in (0.6, 0.6+\delta) | G]}{0.5 Pr[X \in (0.6, 0.6+\delta) | G] + 0.5 Pr[X \in (0.6, 0.6+\delta) | B]} = \frac{e^{-0.6}\delta}{e^{-0.6}\delta + (0.8)^{-1} e^{-(0.8)^{-1} 0.6}\delta} \approx 0.488.$$

$$(b) \quad E[\text{ lifespan of other bulb }] = p \times 1 + (1-p) \times 0.8 \approx 0.9.$$

Common Mistakes

- $\Omega = \{1, 2, 3\}$. Define X, Y with $cov(X, Y) = 0$ and X, Y not independent.
  Let $X = 0, Y = 1$. No: They are independent.
  Let $X(1) = -1, X(2) = 0, X(3) = 1, Y(1) = 0, Y(2) = 1, Y(3) = 0$.
- $3 \times 3.5 = 12.5$. No.
- $E[X^2] = E[X]^2$. No.
- $X = B(n, p) \implies var(X) = n^2 p(1-p)$. No.
- $E[X] = E[X|A] + E[X|\bar{A}]$. No.
- $\sum_{n=0}^{\infty} a^n = 1/a$. No.
- CS70 is difficult. No.
- I will do poorly on the final. No.

Thanks and Best Wishes!