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Induction and Recursion

Couple of more induction proofs.

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Induction: Some quibbles.

What did you learn in 61A?

Induction and Recursion

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Stable Marriage.

Some quibbles.

The induction principle works on the natural numbers.

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Proves statements of form: $\forall n \in \mathbb{N}, P(n)$.

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In some sense, the natural numbers.

Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

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Slight differences: showed for all $n \geq 16$ that $\wedge_{i=4}^{n-1} P(i) \implies P(n)$.

Strengthening: need to...

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

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Need to prove "$S_{k+1} \leq 2 - \frac{1}{(k+2)^2}$".

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"$S_k \leq 2 - \frac{1}{(k+1)^2}$" $\implies$ "$S_{k+1} \leq 2$"
Induction step works! No! Not the same statement!!!!
Need to prove "$S_{k+1} \leq 2 - \frac{1}{(k+2)^2}$".

Strengthening: need to...

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Base: $P(1)$. $1 \leq 2$.
Ind Step: $\sum_{i=1}^{k} \frac{1}{i^2} \leq 2$.

$$\sum_{i=1}^{k+1} \frac{1}{i^2}$$
$$= \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2}.$$
$$\leq 2 + \frac{1}{(k+1)^2}$$

Uh oh?

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Darn!!!

Strenthening: how?

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n)$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

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Proof:

Strenthening: how?

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Ind hyp: $P(k)$

Strenthening: how?

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Prove: $P(k+1)$

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Prove: $P(k+1)$ – "$S_{k+1} \leq 2 - f(k+1)$"

$$S(k+1) = S_k + \frac{1}{(k+1)^2}$$

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Maybe a linearly decreasing function to keep positive?

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Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$.

Stable Matching Problem

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- n candidates and n jobs.

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- Each job has a ranked preference list of candidates.

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Stable Matching Problem

- *n* candidates and *n* jobs.
- Each job has a ranked preference list of candidates.
- Each candidate has a ranked preference list of jobs.

How should they be matched?

Count the ways..

- Maximize total satisfaction.

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Count the ways..

- Maximize total satisfaction.
- Maximize number of first choices.
- Maximize worse off.
- Minimize difference between preference ranks.

The best laid plans..

Consider the pairs..

- (Anthony) Davis and Pelicans
- (Lonzo) Ball and Lakers

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Davis prefers the Lakers.

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Davis prefers the Lakers.

Lakers prefer Davis.

The best laid plans..

Consider the pairs..

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Uh..oh.

The best laid plans..

Consider the pairs..

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Lakers prefer Davis.

Uh..oh. Sad Lonzo and Pelicans.

So..

Produce a pairing where there are no crazy moves!

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Definition: A **pairing** is disjoint set of n job-candidate pairs.

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Example: A pairing $S = \{(Lakers, Ball); (Pelicans, Davis)\}$.

So..

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Definition: A **rogue couple** b, g^* for a pairing S:
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Example: Davis and Lakers are a rogue couple in S.

A stable pairing??

Given a set of preferences.

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Is there a stable pairing?
How does one find it?

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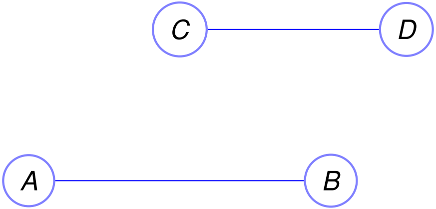
Consider a single type version: stable roommates.

A	B	C	D
B	C	A	D
C	A	B	D
D	A	B	C



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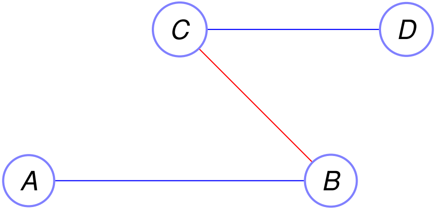

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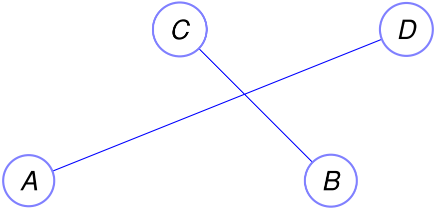

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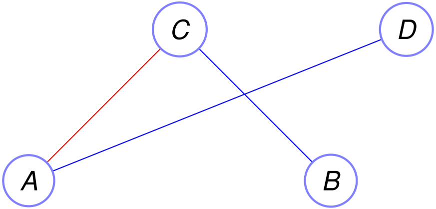

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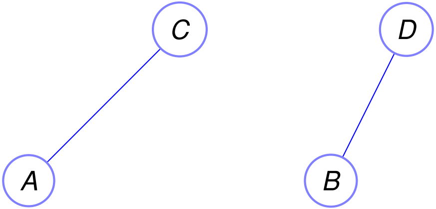

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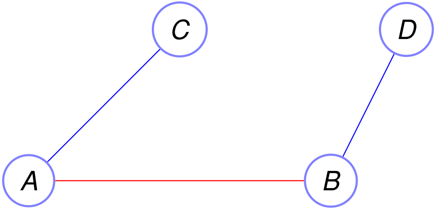

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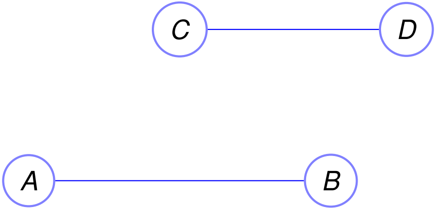
B	C	A	D
C	A	B	D
D	A	B	C



A stable pairing??

Given a set of preferences.

Is there a stable pairing?
How does one find it?

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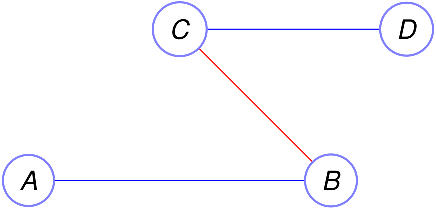

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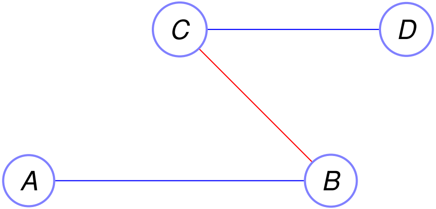

The Propose and Reject Algorithm.

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Each Day:

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Do jobs or candidates do “better”?

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Do jobs or candidates do "better"?

Example.

Jobs			
A	1	2	3
B	1	2	3
C	2	1	3

Candidates			
1	C	A	B
2	A	B	C
3	A	C	B

Example.

	Jobs		
A	1	2	3
B	1	2	3
C	2	1	3

	Candidates		
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1					
2					
3					

Example.

	Jobs		
A	1	2	3
B	1	2	3
C	2	1	3

	Candidates		
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, B				
2	C				
3					

Example.

Jobs			
A	1	2	3
B	~~1~~	2	3
C	2	1	3

Candidates			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~				
2	C				
3					

Example.

Jobs			
A	1	2	3
B	~~1~~	2	3
C	2	1	3

Candidates			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A			
2	C	B, C			
3					

Example.

Jobs			
A	1	2	3
B	~~1~~ X	2	3
C	~~2~~ X	1	3

Candidates			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~ X	A			
2	C	B, ~~C~~ X			
3					

Example.

Jobs			
A	1	2	3
B	~~1~~	2	3
C	~~2~~	1	3

Candidates			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A	A , C		
2	C	B, ~~C~~	B		
3					

Example.

Jobs			
A	~~1~~	2	3
B	~~1~~	2	3
C	~~2~~	1	3

Candidates			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A	~~A~~ , C		
2	C	B, ~~C~~	B		
3					

Example.

Jobs			
A	~~1~~	2	3
B	~~1~~	2	3
C	~~2~~	1	3

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1	C	A	B
2	A	B	C
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	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A	~~A~~ , C	C	
2	C	B, ~~C~~	B	A,B	
3					

Example.

Jobs			
A	~~1~~	2	3
B	~~1~~	~~2~~	3
C	~~2~~	1	3

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1	C	A	B
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	Day 1	Day 2	Day 3	Day 4	Day 5
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2	C	B, ~~C~~	B	A, ~~B~~	
3					

Example.

Jobs			
A	~~1~~	2	3
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1	A, ~~B~~	A	~~A~~ , C	C	C
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3					B

Example.

Jobs			
A	~~1~~	2	3
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	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A	~~A~~ , C	C	C
2	C	B, ~~C~~	B	A, ~~B~~	A
3					B

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Every non-terminated day a job **crossed** an item off the list.

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Terminates in $\leq n^2$ steps!

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Proof Idea: She can always keep the previous job on the string.

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$P(k)$- - "job on g's string is at least as good as b on day $t+k$"

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Candidate g can choose b',

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Candidate g can choose b', or do better with another job, b''

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That is,

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And by principle of induction, lemma holds for every day after t.

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$P(k)$- - "job on g's string is at least as good as b on day $t+k$"

$P(0)$– true. Candidate has b on string.

Assume $P(k)$. Let b' be job **on string** on day $t+k$.

On day $t+k+1$, job b' comes back.
Candidate g can choose b', or do better with another job, b''

That is, $b' \leq b$ by induction hypothesis.
And b'' is better than b' by algorithm.
$\implies$ Candidate does at least as well as with b.

$P(k) \implies P(k+1)$.
And by principle of induction, lemma holds for every day after t. □

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Lemma: Every job is matched at end.
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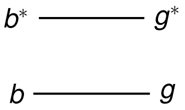

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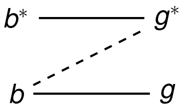

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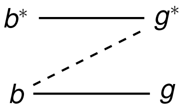

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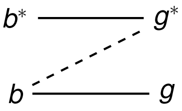

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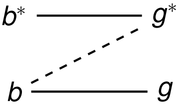

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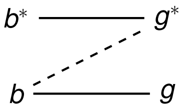

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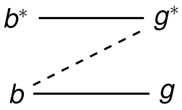

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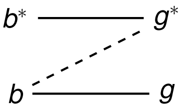

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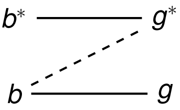

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Which is optimal for A?

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Which is optimal for A? S

Understanding Optimality: by example.

A:	1,2	1:	A,B
B:	1,2	2:	B,A

Consider pairing: $(A,1),(B,2)$.

Stable? Yes.

Optimal for B?
Notice: only one stable pairing.
So this is the best B can do in a stable pairing.
So optimal for B.

Also optimal for A, 1 and 2. Also pessimal for A,B,1 and 2.

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Job Propose and Candidate Reject is optimal!

For jobs?

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For jobs? For candidates?

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Theorem: Job Propose and Reject produces a job-optimal pairing.

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Structural statement: Job optimality

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Structural statement: Job optimality $\implies$ Candidate pessimality.

Quick Questions.

How does one make it better for candidates?

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Propose and Reject - stable matching algorithm. One side proposes.

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Jobs Propose $\implies$ job optimal.

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Propose and Reject - stable matching algorithm. One side proposes.

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Candidates propose.

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Jobs Propose $\implies$ job optimal.

Candidates propose. $\implies$ optimal for candidates.

Residency Matching..

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The method was used to match residents to hospitals.

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Hospital optimal....

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Another variation: couples.

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Proofs carefully use definition:

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Proofs carefully use definition:
  Optimality proof:
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