Today.

Quick review.

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Finish Graphs (maybe.)

Proof of "handshake" lemma.

Lemma: The sum of degrees is $2|E|$, for a graph $G = (V, E)$.
What's true?

(A) The number of edge-vertex incidences for an edge e is 2.
(B) The total number of edge-vertex incidences is $|V|$.
(C) The total number of edge-vertex incidences is $2|E|$.
(D) The number of edge-vertex incidences for a vertex v is its degree.
(E) The sum of degrees is $2|E|$.
(F) Total number of edge-vertex incidences is sum of vertex degrees.

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(E) The sum of degrees is $2|E|$.
(F) Total number of edge-vertex incidences is sum of vertex degrees.

(B) is false. The others are statements in the proof.

Poll: Euler concepts.

A graph is Euleurian if it is connected and has even degree.

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(A) There is no Hotel California in this graph.
(B) Walking on unused edges, starting at v, eventually return to v.
(C) Removing a tour leaves a graph of even degree.
(D) Removing a tour leaves a connected graph.
(E) Remove set of edges E' in connected graph, connected component is incident to edge in E'
(F) A tour connecting a set of connected components, each with a Eulerian tour is really cool! This implies the graph is Eulerian.

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Only (C) is false. The rest are steps in the proof.

Lecture 6.

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Euler's Formula.

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Planar Six and then Five Color theorem.

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Types of graphs.

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- Complete Graphs.
- Trees (a little more.)
- Hypercubes.

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A graph that can be drawn in the plane without edge crossings.

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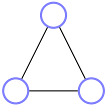

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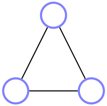

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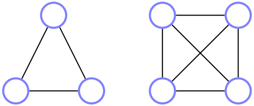

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Four node complete?

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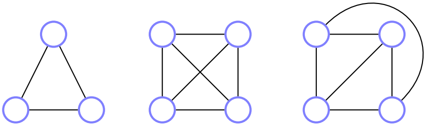

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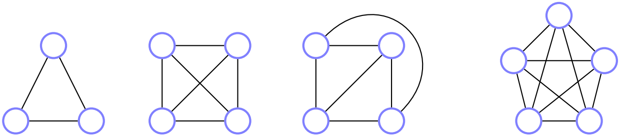

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Five node complete or K_5 ?

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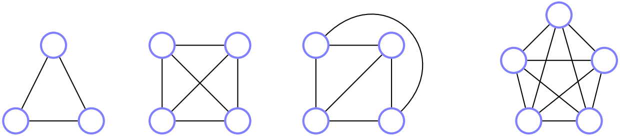

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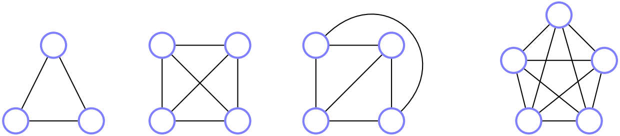

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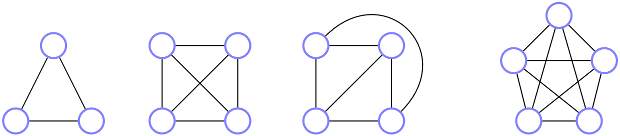

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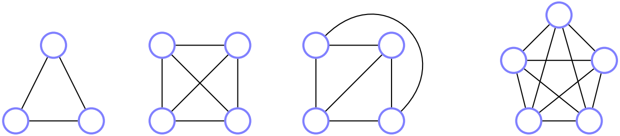

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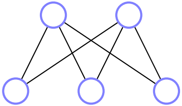

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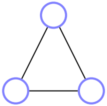 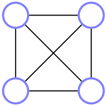 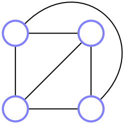 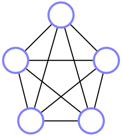

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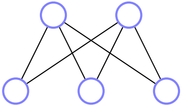

Two to three nodes, bipartite?

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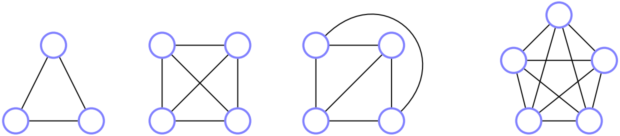

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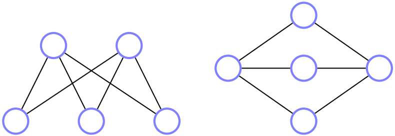

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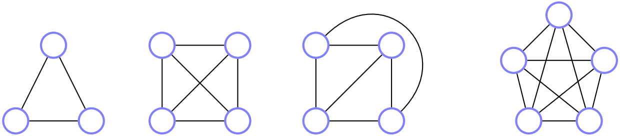

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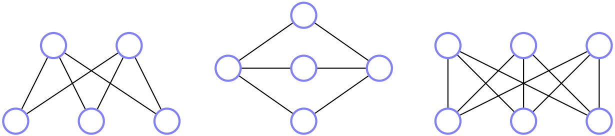

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Three to three nodes, complete/bipartite or $K_{3,3}$.

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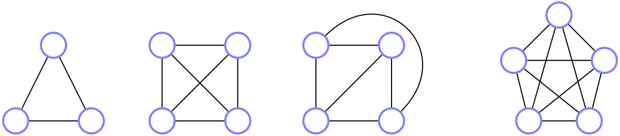

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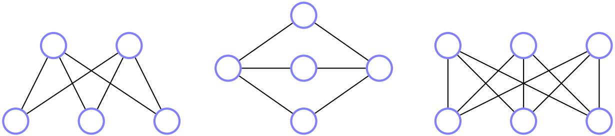

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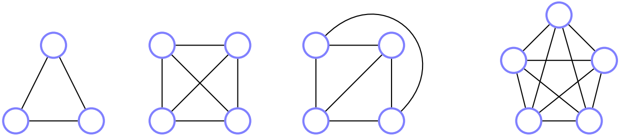

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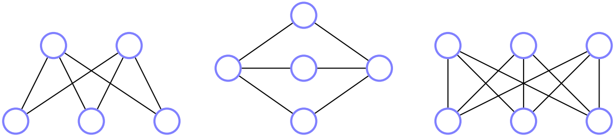

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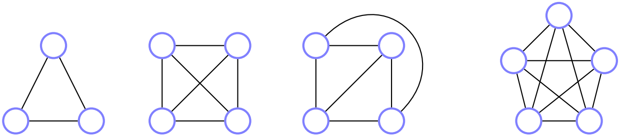

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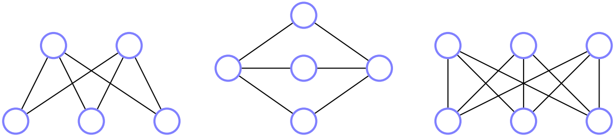

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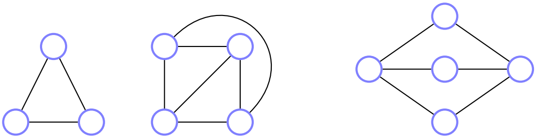

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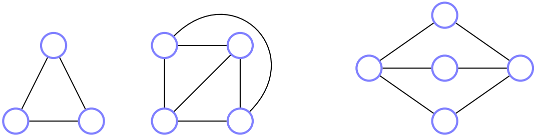
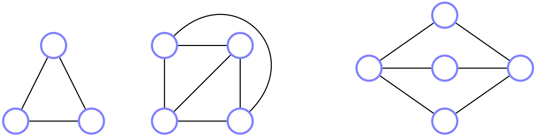


Faces: connected regions of the plane.

Euler's Formula.



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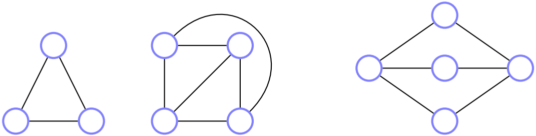
How many faces for

Euler's Formula.



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How many faces for
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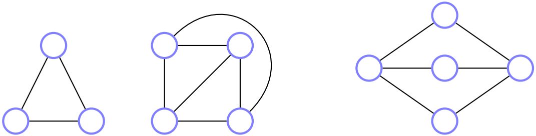

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How many faces for

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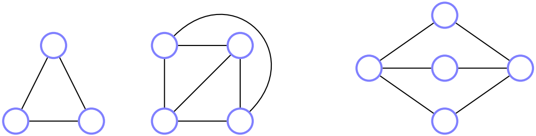
Euler's Formula.



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How many faces for

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complete on four vertices or K_4?

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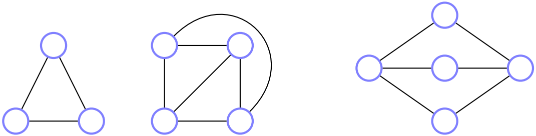

Faces: connected regions of the plane.

How many faces for

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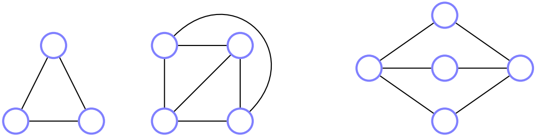

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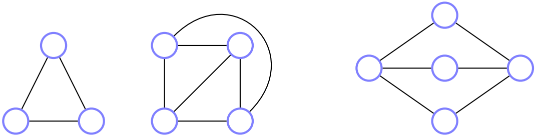

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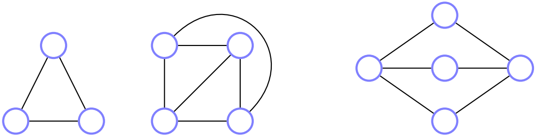

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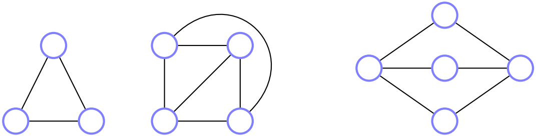

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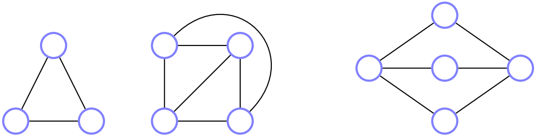

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Euler's Formula: Connected planar graph has $v + f = e + 2$.

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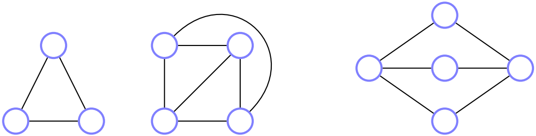

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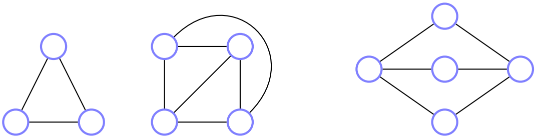

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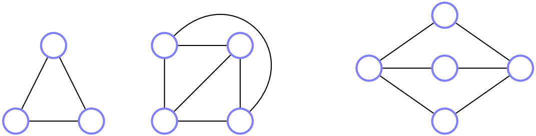

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Triangle: $3 + 2 = 3 + 2$!

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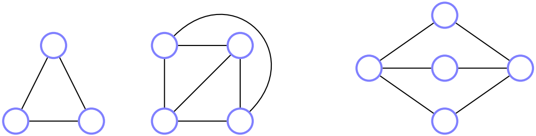

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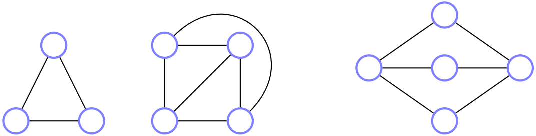

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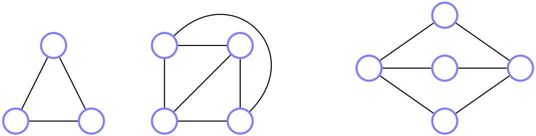
Triangle: $3 + 2 = 3 + 2$!
K_4: $4 + 4 = 6 + 2$!

Euler's Formula.



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$K_{2,3}$:

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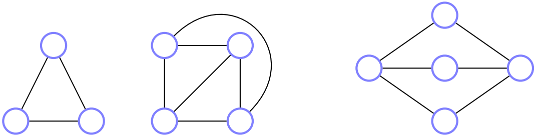

Faces: connected regions of the plane.

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complete on four vertices or K_4? 4

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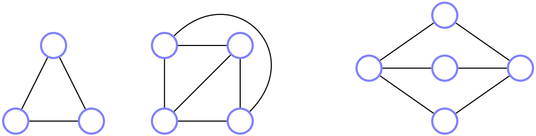

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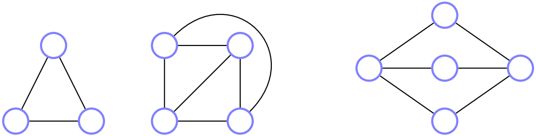

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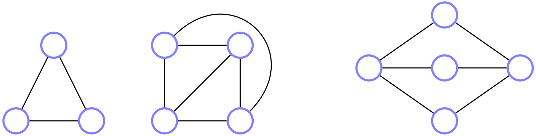

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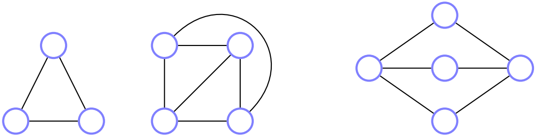

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Euler and Polyhedron.

Greeks knew formula for polyhedron.

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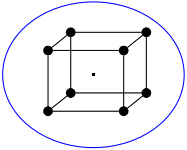

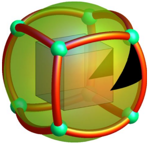

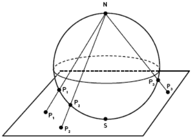

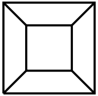

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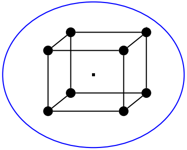

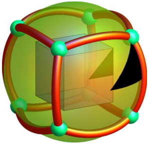

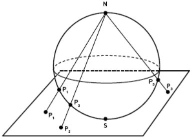

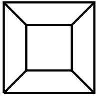

Faces?

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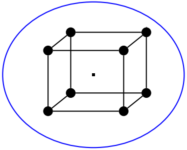

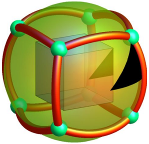

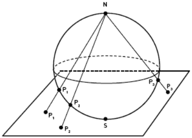

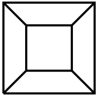

Faces? 6.   Edges?

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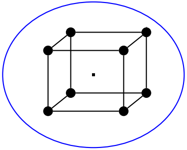
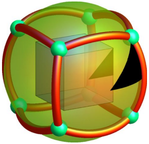
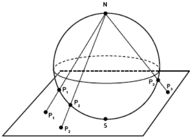

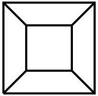

Faces? 6. Edges? 12.

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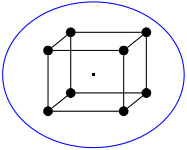

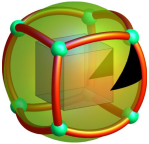

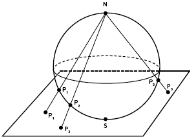

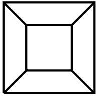

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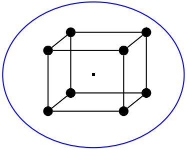

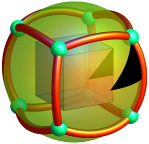

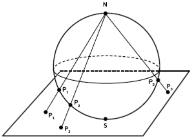

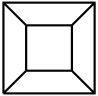

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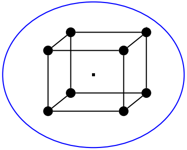
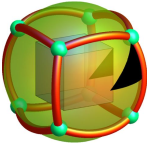
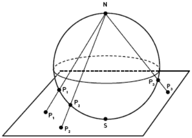

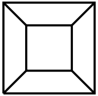

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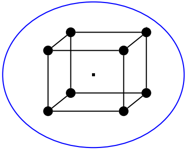
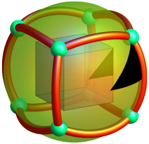
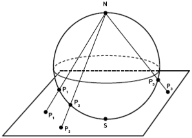

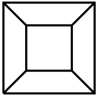

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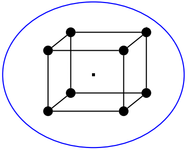

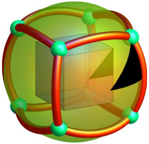

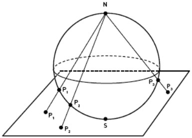

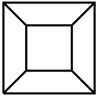

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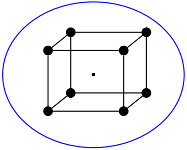
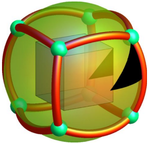
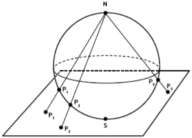

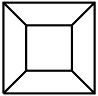

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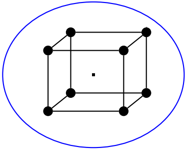
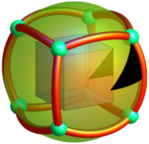
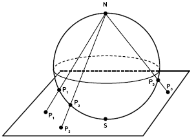

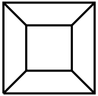

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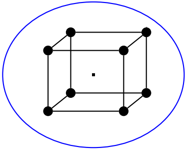
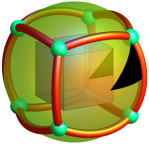
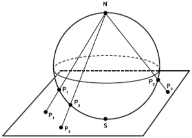

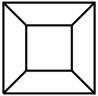

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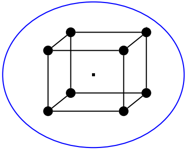

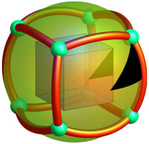

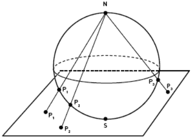

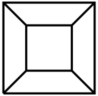

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Polyhedron without holes

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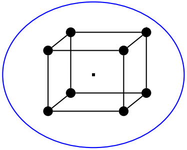

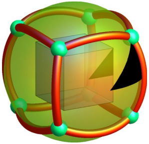

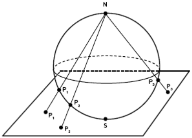

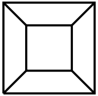

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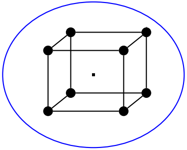
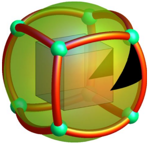
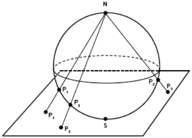

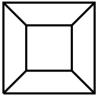

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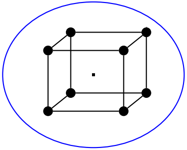
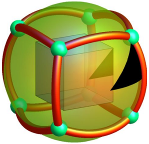
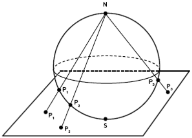

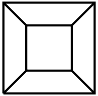

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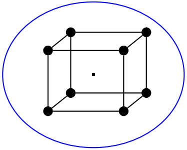
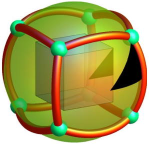
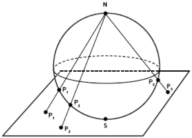

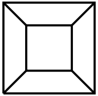

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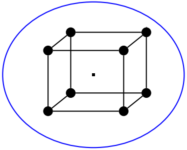
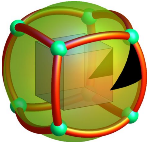
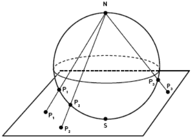

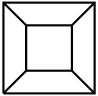

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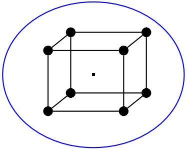

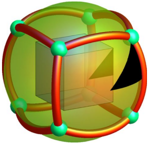

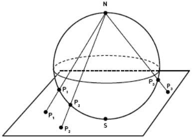

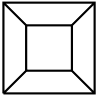

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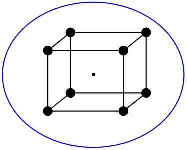
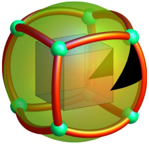
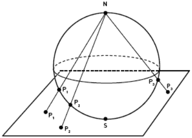

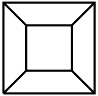

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Project Sphere-N

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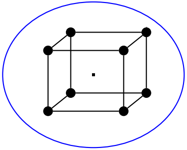
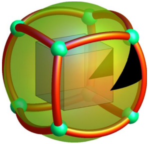
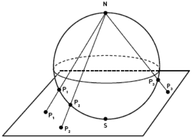

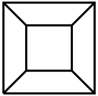

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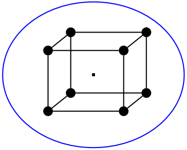 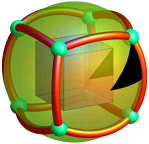 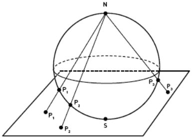 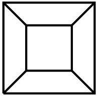

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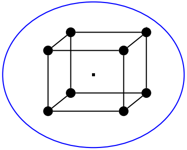
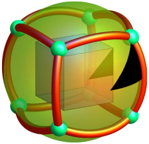
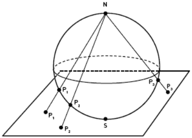

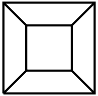

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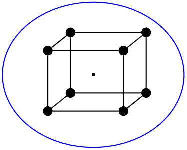 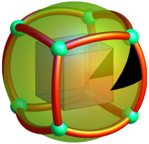 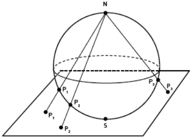

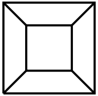

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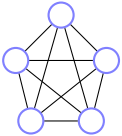
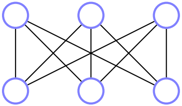
Euler proved formula thousands of years later!

Euler and non-planarity of K_5 and $K_{3,3}$





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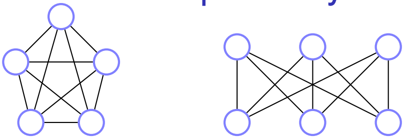

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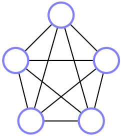

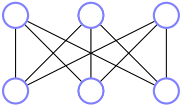

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We consider simple graphs where $v \geq 3$.

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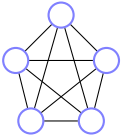

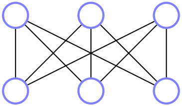

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- Consider Face edge Adjacencies with multiplicities

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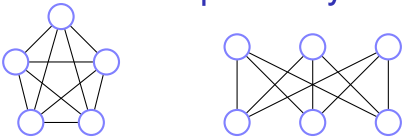

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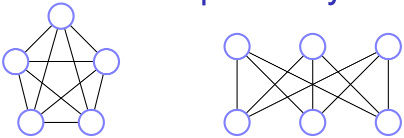

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Consider Face edge Adjacencies with multiplicities

Each face is adjacent to at least three edges($v>2$).

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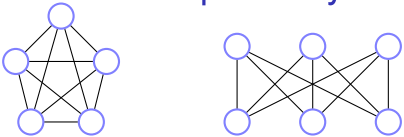

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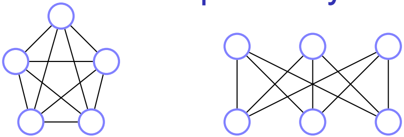

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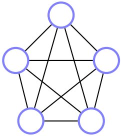

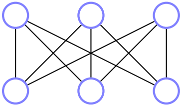

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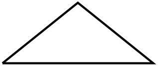

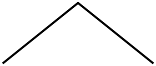

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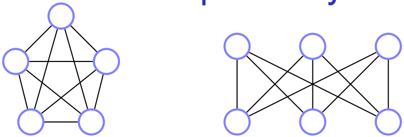

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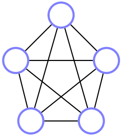

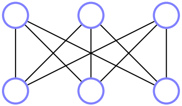

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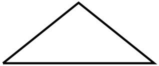
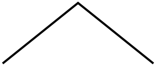
Consider Face edge Adjacencies with multiplicities





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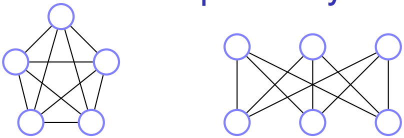

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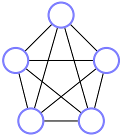

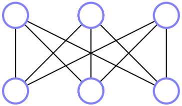

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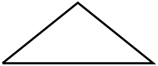

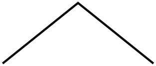

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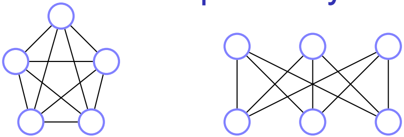

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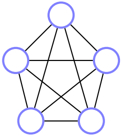

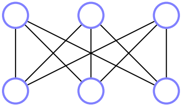

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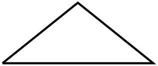

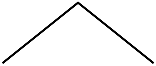

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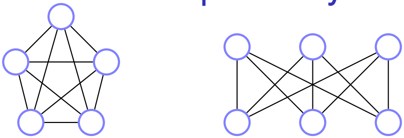

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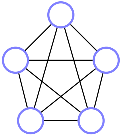
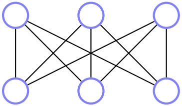

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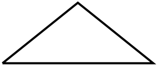
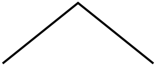

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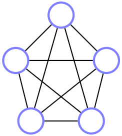

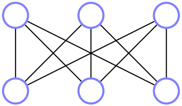

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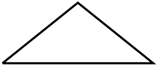

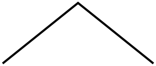

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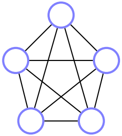

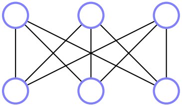

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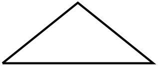

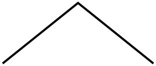

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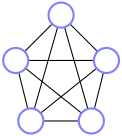
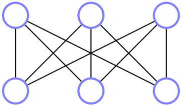
K_5

Euler and non-planarity of K_5 and $K_{3,3}$





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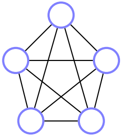
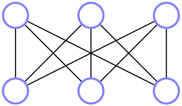
K_5 Edges?

Euler and non-planarity of K_5 and $K_{3,3}$





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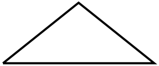

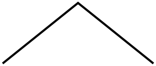

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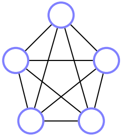

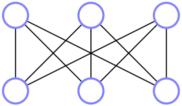

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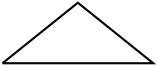

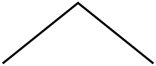

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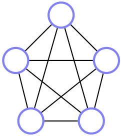
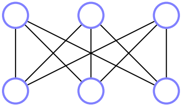
Euler and non-planarity of K_5 and $K_{3,3}$





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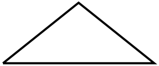

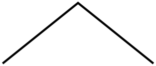

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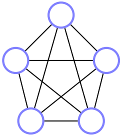

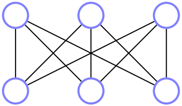

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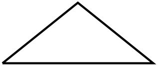

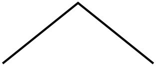

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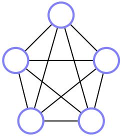

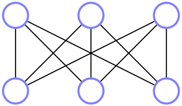

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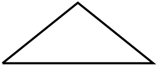

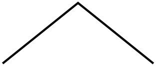

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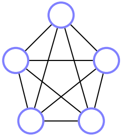
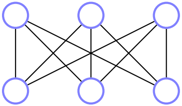

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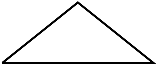
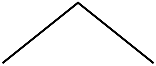
Consider Face edge Adjacencies with multiplicities

Each face is adjacent to at least three edges($v > 2$).
$\geq 3f$ face-edge adjacencies.
Each edge is adjacent to two faces.
$= 2e$ face-edge adjacencies.
$\implies 3f \leq 2e$ for any planar graph with $v > 2$. Or $f \leq \frac{2}{3}e$.

Plug into Euler: $v + \frac{2}{3}e \geq e + 2 \implies e \leq 3v - 6$

K_5 Edges? $e = 4 + 3 + 2 + 1 = 10$. Vertices? $v = 5$.
$10 \not\leq 3(5) - 6 = 9$. $\implies$ K_5 is not planar.

Planar $\implies e \le 3v - 6$. Flow Poll.

Euler's formula: $v + f = e + 2$

Consider graph with > 2 vertices. Understand the following.

(A) Every face is incident to ≥ 3 edges.

(B) Face-edge incidences $\geq 3f$

(C) Every edge is incident (with multiplicity) to 2 faces.

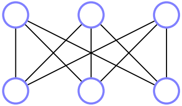
(D) Face edge incidences $= 2e$

(E) $3f \leq$ Face-ege-incidence $= 2e$

(F) $3(e+2-v) <= 2e$

Conclusion: $e <= 3v - 6$

Proving non-planarity for $K_{3,3}$



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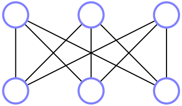

$K_{3,3}$?

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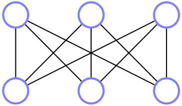

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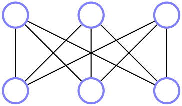

$K_{3,3}$? Edges? 9.

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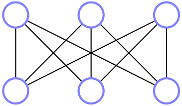

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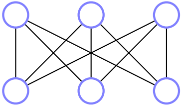

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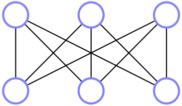

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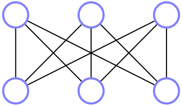

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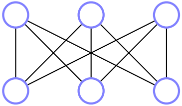

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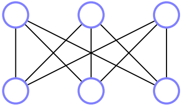

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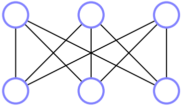

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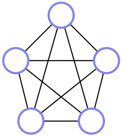
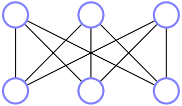
Finish in homework!

Planarity and Euler





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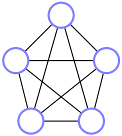

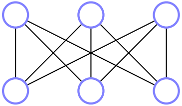

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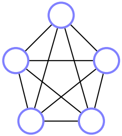

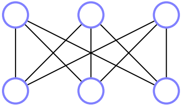

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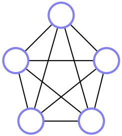

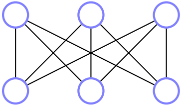

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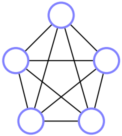

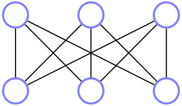

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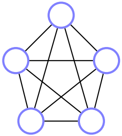

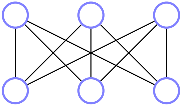

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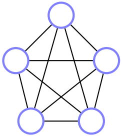

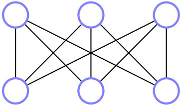

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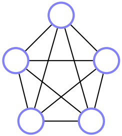

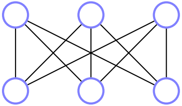

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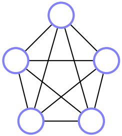

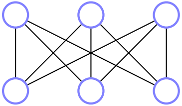

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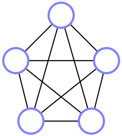

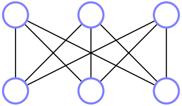

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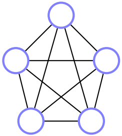
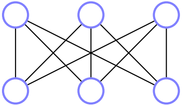
Planarity and Euler





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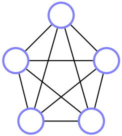

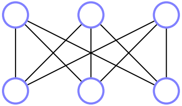

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So......so ...Cool!

Euler's formula.

Euler: Connected planar graph has $v + f = e + 2$.

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Proof:

Euler's formula.

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Proof: Induction on e.

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Proof: Induction on e.
Base:

Euler's formula.

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Proof: Induction on e.
Base: $e = 0$,

Euler's formula.

Euler: Connected planar graph has $v + f = e + 2$.

Proof: Induction on e.
Base: $e = 0$, $v = f = 1$.

Euler's formula.

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Proof: Induction on e.
Base: $e=0$, $v=f=1$.
Induction Step:

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Proof: Induction on e.
Base: $e = 0$, $v = f = 1$.
Induction Step:
  If it is a tree.

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Proof: Induction on e.
Base: $e = 0$, $v = f = 1$.
Induction Step:
If it is a tree. $e = v - 1$, $f = 1$, $v + 1 = (v - 1) + 2$. Yes.

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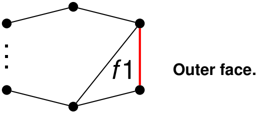

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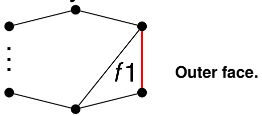

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New graph: v-vertices.

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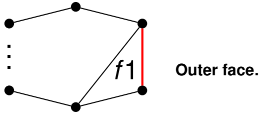

Joins two faces.
New graph: v-vertices. $e - 1$ edges.

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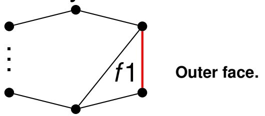

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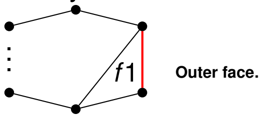

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New graph: v-vertices. $e-1$ edges. $f-1$ faces. Planar.

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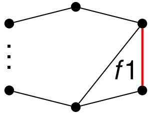
Proof: Induction on e.
Base: $e = 0$, $v = f = 1$.
Induction Step:
If it is a tree. $e = v - 1$, $f = 1$, $v + 1 = (v - 1) + 2$. Yes.
If not a tree.
Find a cycle. Remove edge.



Outer face.

Joins two faces.
New graph: v-vertices. $e - 1$ edges. $f - 1$ faces. Planar.
$v + (f - 1) = (e - 1) + 2$ by induction hypothesis.

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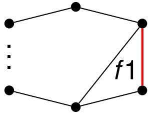

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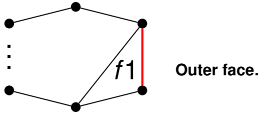

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Quick:
$v + 1 = (v - 1) + 2$, add edge: $f \rightarrow f + 1$, $e \rightarrow e + 1$.

Euler's Proof.Poll.

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Steps/concepts in proof of euler's formula.

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(A) Planar drawing of tree has 1 face.
(B) Tree has $|V| - 1$ edges.
(C) Induction.
(D) face is adjacent to at least 3 edges.
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All are true and relevant to proof.

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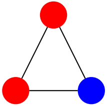

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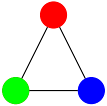

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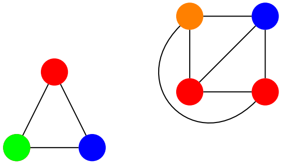

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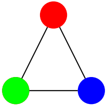

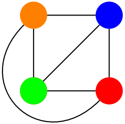

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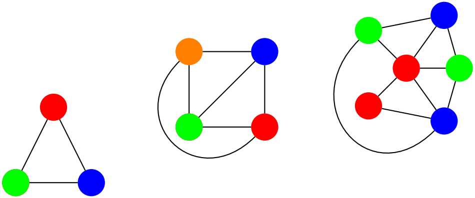

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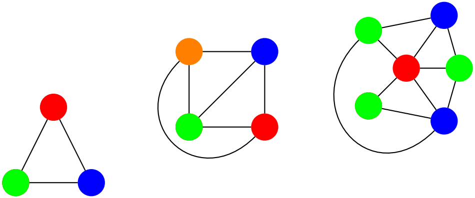

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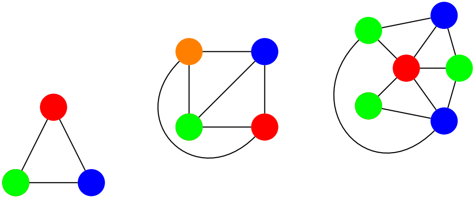

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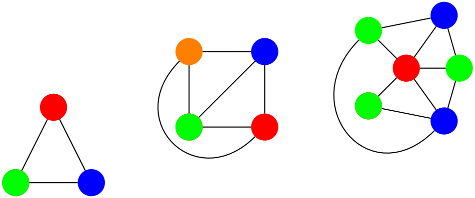

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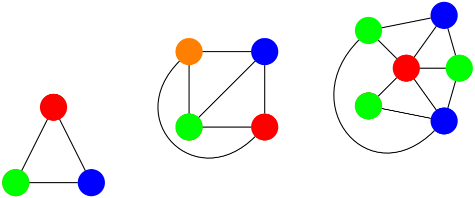

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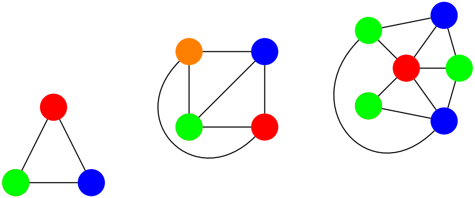

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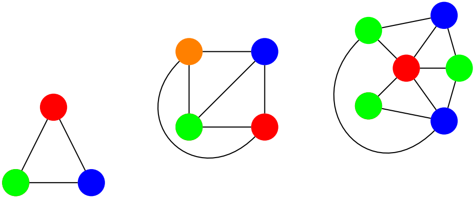

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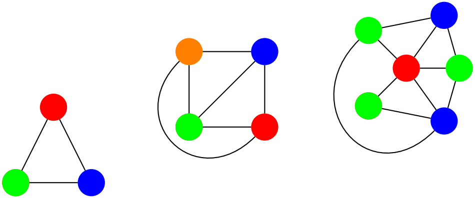

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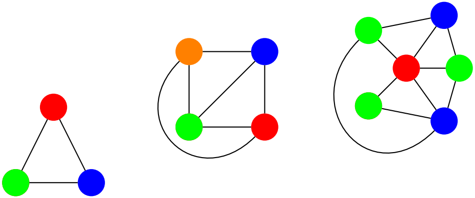

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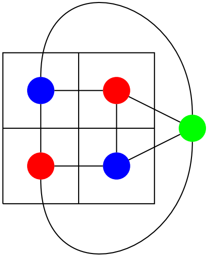
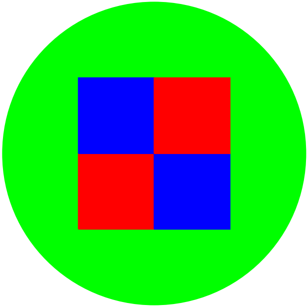
Interesting things to do. Algorithm!

Planar graphs and maps.

Planar graph coloring $\equiv$ map coloring.





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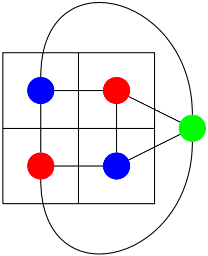

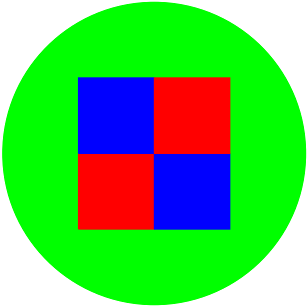

Four color theorem is about planar graphs!

Six color theorem.

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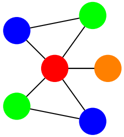
Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.



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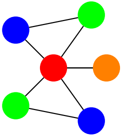

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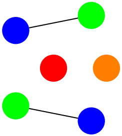

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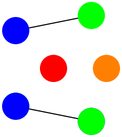

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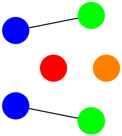

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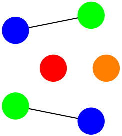

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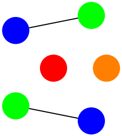

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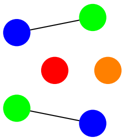

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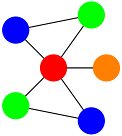

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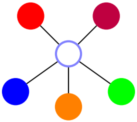

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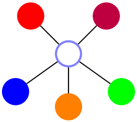

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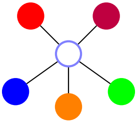

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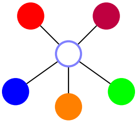

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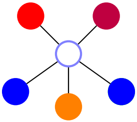

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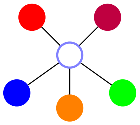

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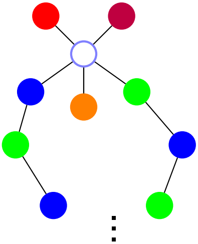

Assume neighbors are colored all differently.
Otherwise one of 5 colors is available. $\Longrightarrow$ Done!
Switch green and blue in green's component.
Done. Unless blue-green path to blue.

Five color theorem

Theorem: Every planar graph can be colored with five colors.

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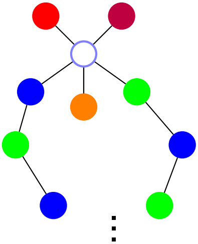

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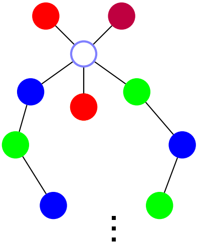

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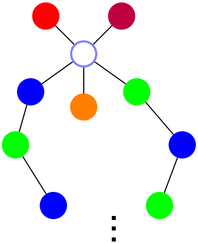

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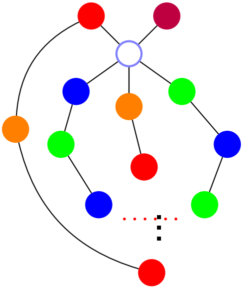

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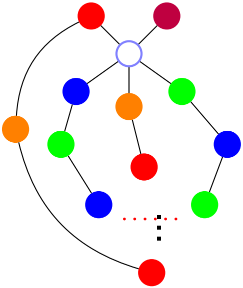

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Planar.

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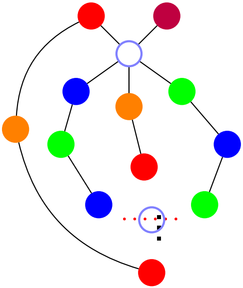

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Planar. $\implies$ paths intersect at a vertex!

Five color theorem

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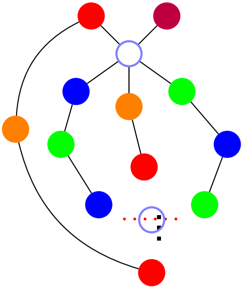

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What color is it?

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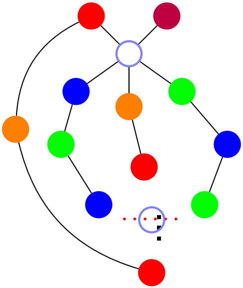

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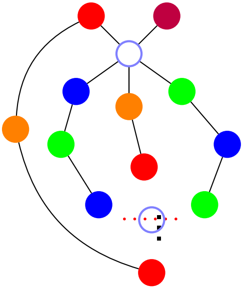

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Planar. $\Longrightarrow$ paths intersect at a vertex!

What color is it?
Must be blue or green to be on that path.

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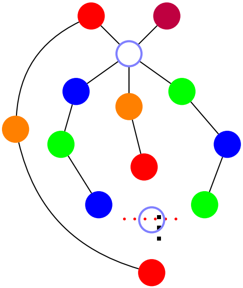

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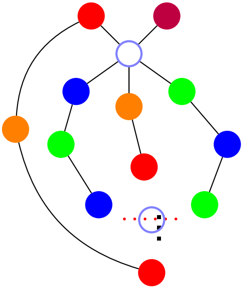

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Planar. $\Longrightarrow$ paths intersect at a vertex!

What color is it?
Must be blue or green to be on that path.
Must be red or orange to be on that path.

Contradiction.

Five color theorem

Theorem: Every planar graph can be colored with five colors.

Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.

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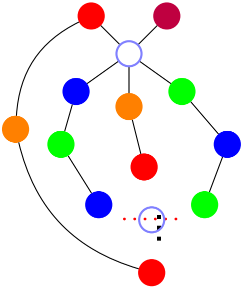

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Planar. $\implies$ paths intersect at a vertex!

What color is it?
Must be blue or green to be on that path.
Must be red or orange to be on that path.

Contradiction. Can recolor one of the neighbors.

Five color theorem

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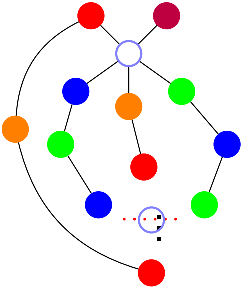

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Planar. $\implies$ paths intersect at a vertex!

What color is it?
Must be blue or green to be on that path.
Must be red or orange to be on that path.

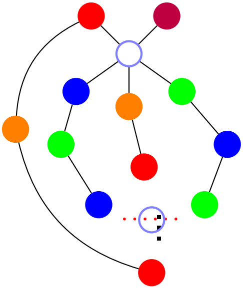
Contradiction. Can recolor one of the neighbors.
Gives an available color for center vertex!

Five color theorem

Theorem: Every planar graph can be colored with five colors.

Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.

Proof: Again with the degree 5 vertex. Again recurse.



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Planar. $\Longrightarrow$ paths intersect at a vertex!

What color is it?
Must be blue or green to be on that path.
Must be red or orange to be on that path.

Contradiction. Can recolor one of the neighbors.
Gives an available color for center vertex! □

5 color theorem. Flow poll.

Steps/ideas in 5-color theorem.
(A) There is a degree 5 vertex cuz Euler.
(B) Take subgraph of first and third colors, recolor first components.
(C) If a third's component is different, switched coloring is good.
(D) Subgraph of second and fourth colors, can recolor, recolor second component.
(G) At least one separate component cuz planarity.
(F) Shared color of five neighbors, done.

5 color theorem. Flow poll.

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All steps in proof!

Four Color Theorem

Four Color Theorem

Theorem: Any planar graph can be colored with four colors.

Four Color Theorem

Theorem: Any planar graph can be colored with four colors.

Proof:

Four Color Theorem

Theorem: Any planar graph can be colored with four colors.

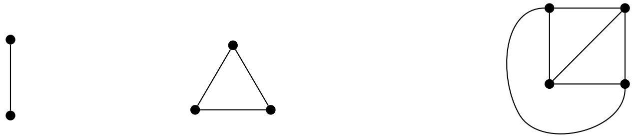
Proof: Not Today!

Four Color Theorem

Theorem: Any planar graph can be colored with four colors.

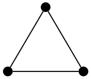
Proof: Not Today!

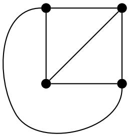
Complete Graph.



K_n complete graph on n vertices.

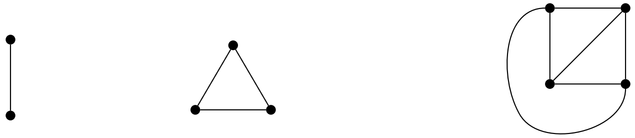
Complete Graph.





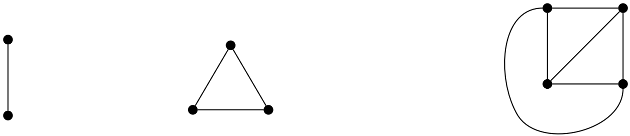
K_n complete graph on n vertices.
All edges are present.

Complete Graph.



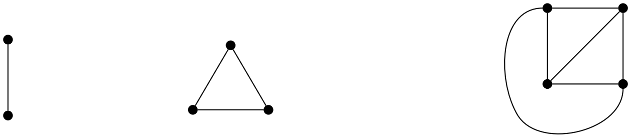
K_n complete graph on n vertices.
All edges are present.
Everyone is my neighbor.

Complete Graph.



K_n complete graph on n vertices.
  All edges are present.
  Everyone is my neighbor.
  Each vertex is adjacent to every other vertex.

Complete Graph.



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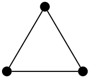

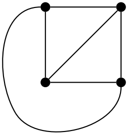

K_n complete graph on n vertices.
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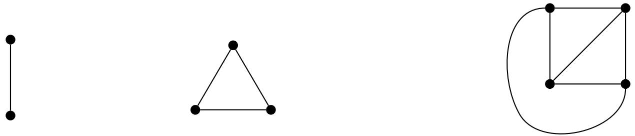
- Everyone is my neighbor.
- Each vertex is adjacent to every other vertex.

How many edges?

Complete Graph.



K_n complete graph on n vertices.

All edges are present.

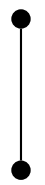
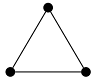
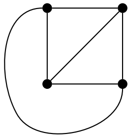
Everyone is my neighbor.

Each vertex is adjacent to every other vertex.

How many edges?

Each vertex is incident to $n-1$ edges.

Complete Graph.

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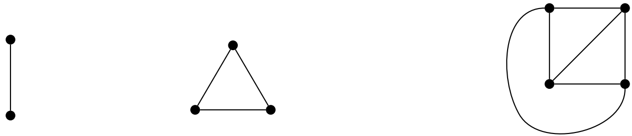
Sum of degrees is $n(n-1)$

Complete Graph.



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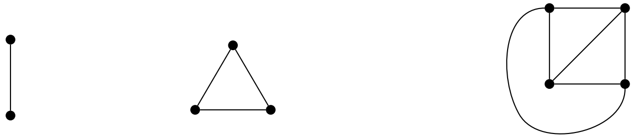
All edges are present.
Everyone is my neighbor.
Each vertex is adjacent to every other vertex.

How many edges?
Each vertex is incident to $n-1$ edges.
Sum of degrees is $n(n-1) = 2|E|$

Complete Graph.



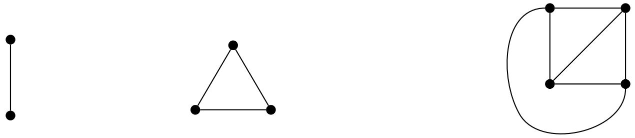
K_n complete graph on n vertices.

All edges are present.
Everyone is my neighbor.
Each vertex is adjacent to every other vertex.

How many edges?

Each vertex is incident to $n-1$ edges.
Sum of degrees is $n(n-1) = 2|E|$
$\implies$ Number of edges is $n(n-1)/2$.

Complete Graph.



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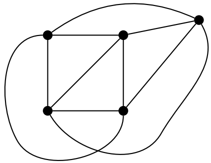
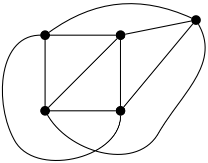
K_4 and K_5



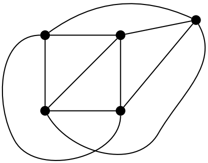
K_5 is not planar.

K_4 and K_5



K_5 is not planar.
Cannot be drawn in the plane without an edge crossing!

K_4 and K_5

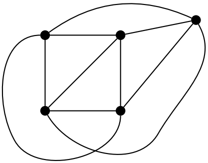


K_5 is not planar.

Cannot be drawn in the plane without an edge crossing!

Prove it!

K_4 and K_5



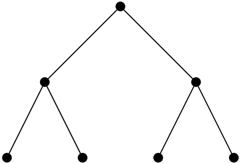
K_5 is not planar.

Cannot be drawn in the plane without an edge crossing!

Prove it! We did!

Tree's fall apart.

Thm: There is one vertex whose removal disconnects $|V|/2$ nodes from each other.



Idea of proof.

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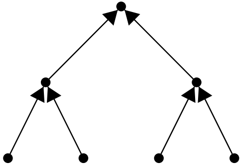
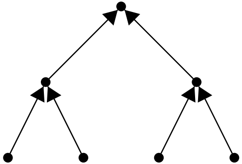


Idea of proof.
Point edge toward bigger side.

Tree's fall apart.

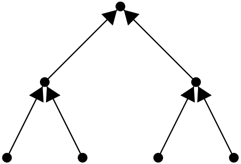
Thm: There is one vertex whose removal disconnects $|V|/2$ nodes from each other.



Idea of proof.
Point edge toward bigger side.
Remove center node:

Tree's fall apart.

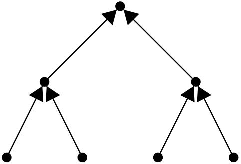
Thm: There is one vertex whose removal disconnects $|V|/2$ nodes from each other.



Idea of proof.
Point edge toward bigger side.
Remove center node: node with no outgoing arc. (Hotel California.)

Tree's fall apart.

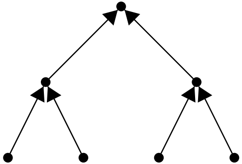
Thm: There is one vertex whose removal disconnects $|V|/2$ nodes from each other.



Idea of proof.
Point edge toward bigger side.
Remove center node: node with no outgoing arc. (Hotel California.)
All the neighbors in components that are smaller than $|V|/2$.

Tree's fall apart.

Thm: There is one vertex whose removal disconnects $|V|/2$ nodes from each other.



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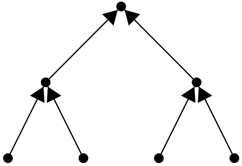

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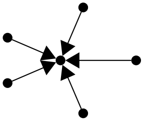

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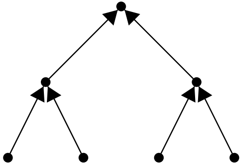

Idea of proof.
Point edge toward bigger side.
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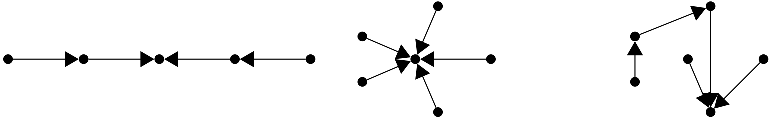
All the neighbors in components that are smaller than $|V|/2$.



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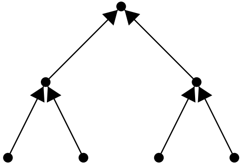

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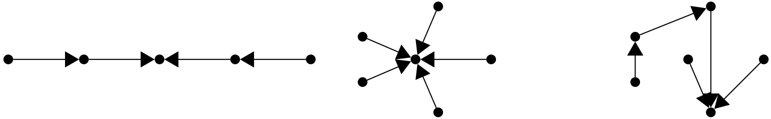

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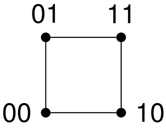

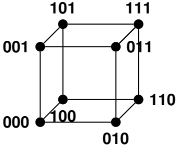

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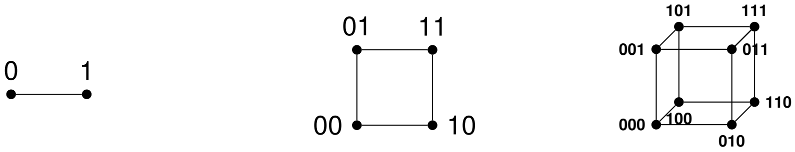

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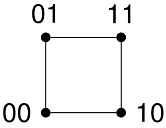

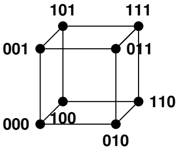

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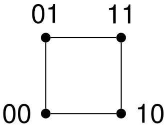

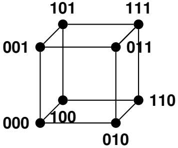

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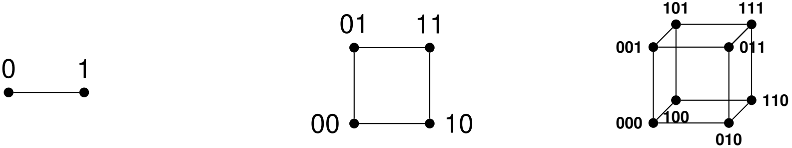

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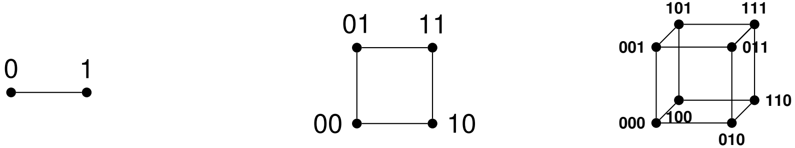

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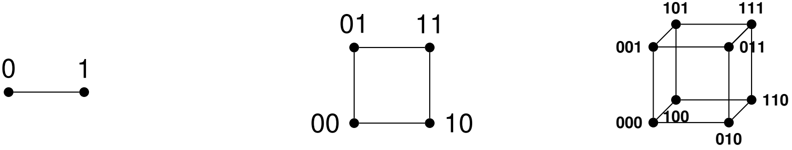

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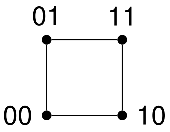

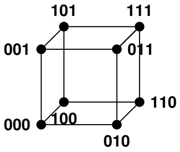

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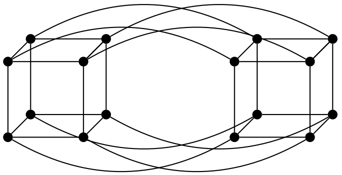

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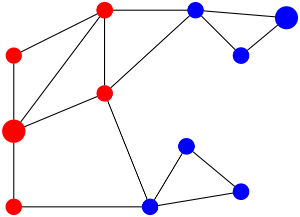
Restatement: for any cut in the hypercube, the number of cut edges is at least the size of the small side.

Cuts in graphs.



S is red, $V - S$ is blue.

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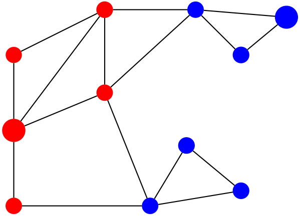

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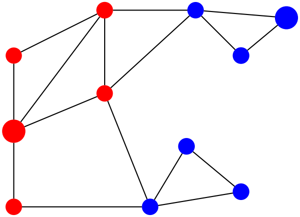

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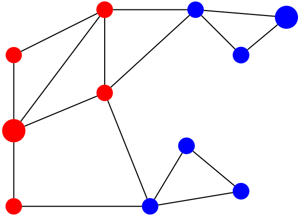

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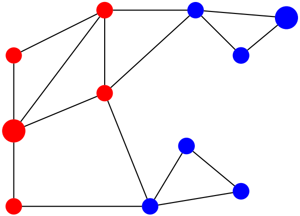

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Hypercube: any cut that cuts off x nodes has $\geq x$ edges.

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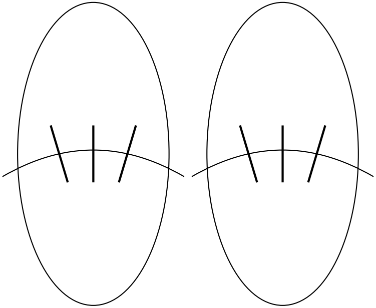

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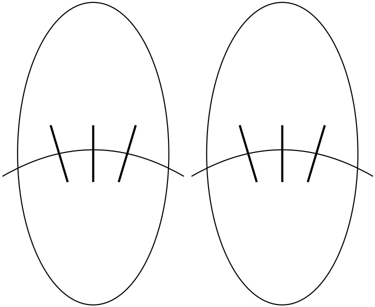

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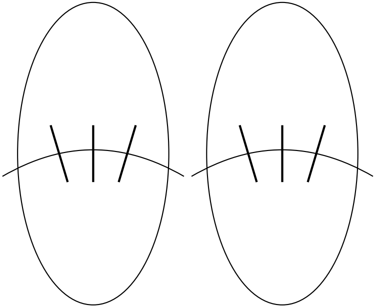

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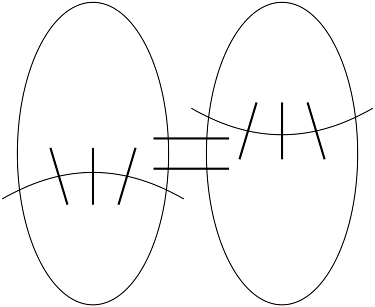

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Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

Proof: Induction Step.
Recursive definition:
$H_0 = (V_0, E_0)$,$H_1 = (V_1, E_1)$, edges E_x that connect them.
$H = (V_0 \cup V_1, E_0 \cup E_1 \cup E_x)$
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Case 1: $|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2$
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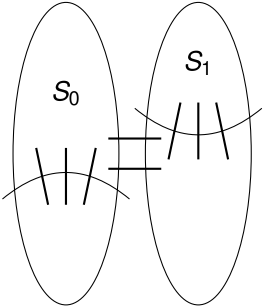

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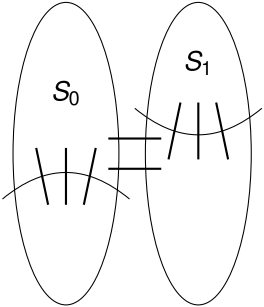

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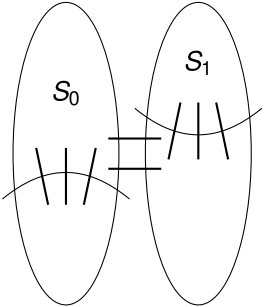

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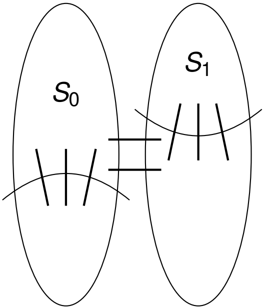

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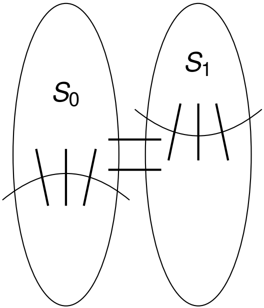

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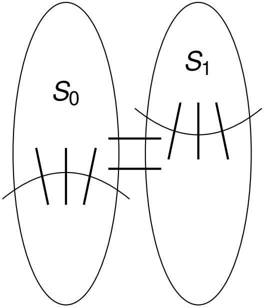

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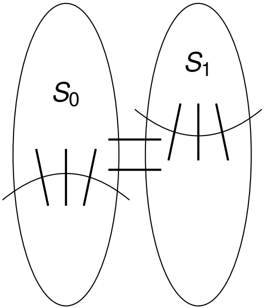

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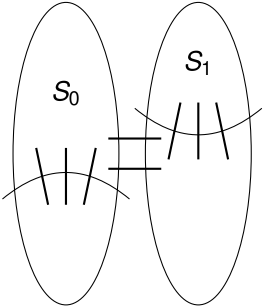

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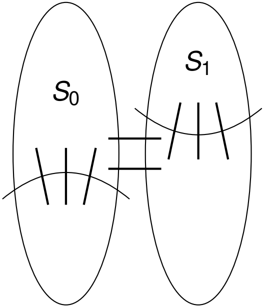

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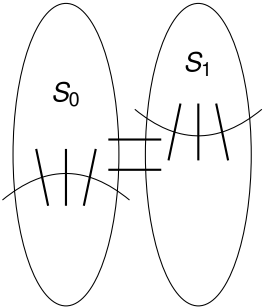

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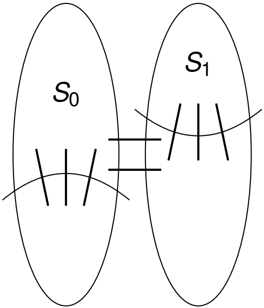

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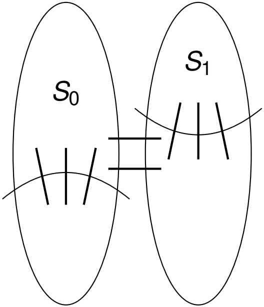

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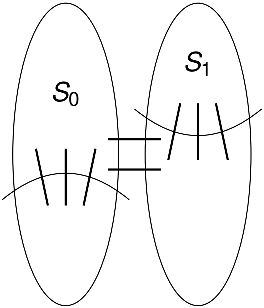

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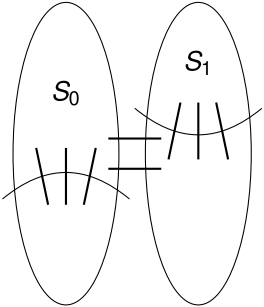

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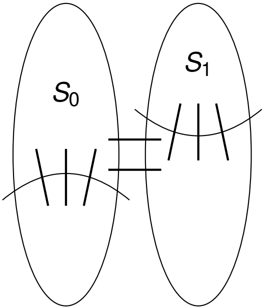

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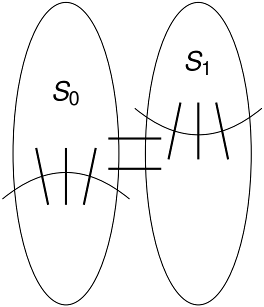

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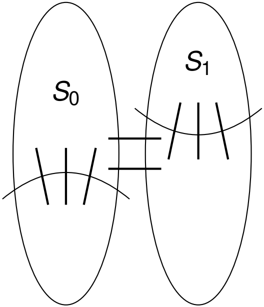

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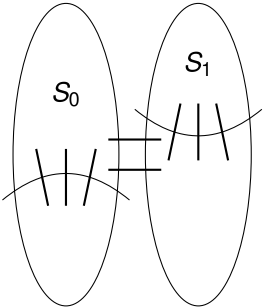

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Also, case 3 where $|S_1| \geq |V|/2$ is symmetric.

Hypercubes and Boolean Functions.

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Graphs:
 Trees: sparsest connected.
 Complete:densest
 Hypercube: middle.

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Planar graphs: $e \leq 3v=6$.
Count face-edge incidences to get $2e \leq 3f$.
Replace f in Euler.
Coloring:
degree d vertex can be colored if $d+1$ colors.
Small degree vertex in planar graph: 6 color theorem.
Recolor separate and planarity: 5 color theorem.
Graphs:
Trees: sparsest connected.
Complete:densest
Hypercube: middle.

Have a nice weekend!