Lecture 7. Outline.

1. Isoperimetric inequality for hypercube.
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2. Modular Arithmetic.
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3. Inverses for Modular Arithmetic: Greatest Common Divisor.
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4. Euclid’s GCD Algorithm.
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1. Isoperimetric inequality for hypercube.
2. Modular Arithmetic.
   Clock Math!!!
3. Inverses for Modular Arithmetic: Greatest Common Divisor.
   Division!!!
4. Euclid’s GCD Algorithm.
   A little tricky here!
Isoperimetry.

For 3-space:

Surface Area: $4\pi r^2$, Volume: $\frac{4}{3}\pi r^3$. Ratio: $\frac{1}{\frac{4}{3}r} = \Theta(\frac{V - 1}{3})$.

Graphical Analog: Cut into two pieces and find ratio of edges/vertices on small side.

Tree: $\Theta(\frac{1}{|V|})$.

Hypercube: $\Theta(1)$.

Surface Area is roughly at least the volume!
Isoperimetry.

For 3-space:

The sphere minimizes surface area to volume.
Isoperimetry.

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Ratio: $1/3r = \Theta(V^{-1/3})$. 
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Surface Area is roughly at least the volume!
A 0-dimensional hypercube is a node labelled with the empty string of bits.
Recursive Definition.

A 0-dimensional hypercube is a node labelled with the empty string of bits.

An $n$-dimensional hypercube consists of a 0-subcube (1-subcube) which is a $n - 1$-dimensional hypercube with nodes labelled $0x$ ($1x$) with the additional edges $(0x, 1x)$. 
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An $n$-dimensional hypercube consists of a 0-subcube (1-subcube) which is a $n-1$-dimensional hypercube with nodes labelled $0x$ ($1x$) with the additional edges $(0x, 1x)$.
Hypercube: Can’t cut me!
Thm: Any subset $S$ of the hypercube where $|S| \leq |V|/2$ has $\geq |S|$ edges connecting it to $V - S$;
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Terminology:
**Thm:** Any subset $S$ of the hypercube where $|S| \leq |V|/2$ has $\geq |S|$ edges connecting it to $V - S$; $|E \cap S \times (V - S)| \geq |S|$

Terminology:

$(S, V - S)$ is cut.
**Thm:** Any subset $S$ of the hypercube where $|S| \leq |V|/2$ has $\geq |S|$ edges connecting it to $V - S$; $|E \cap S \times (V - S)| \geq |S|$

**Terminology:**
- $(S, V - S)$ is cut.
- $(E \cap S \times (V - S))$ - cut edges.
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Terminology:

$(S, V - S)$ is cut.

$(E \cap S \times (V - S))$ - cut edges.

Restatement: for any cut in the hypercube, the number of cut edges is at least the size of the small side.
Cuts in graphs.

$S$ is red, $V - S$ is blue.
Cuts in graphs.

$S$ is red, $V - S$ is blue.

What is size of cut?
Cuts in graphs.

$S$ is red, $V - S$ is blue.

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Number of edges between red and blue.
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Number of edges between red and blue. 4.
Cuts in graphs.

$S$ is red, $V - S$ is blue.

What is size of cut?

Number of edges between red and blue. 4.

Hypercube: any cut that cuts off $x$ nodes has $\geq x$ edges.
Proof of Large Cuts.

**Thm:** For any cut \( (S, V - S) \) in the hypercube, the number of cut edges is at least the size of the small side.

**Proof:**
Proof of Large Cuts.

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

**Proof:**
Base Case: \(n = 1\)
Thm: For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

Proof:
Base Case: \(n = 1\) \(V = \{0,1\}\).
Thm: For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

Proof:
Base Case: \(n = 1\ V = \{0, 1\}\. 
\quad S = \{0\} \text{ has one edge leaving.}
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Base Case: \(n = 1\) \(V = \{0,1\}\).
- \(S = \{0\}\) has one edge leaving. \(|S| = \phi\) has 0.
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Induction Step Idea

**Thm:** For any cut \((S, V – S)\) in the hypercube, the number of cut edges is at least the size of the small side.
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**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

Use recursive definition into two subcubes.
Induction Step Idea

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

Use recursive definition into two subcubes.

Two cubes connected by edges.
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Use recursive definition into two subcubes.

Two cubes connected by edges.

Case 1: Count edges inside subcube inductively.
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

Use recursive definition into two subcubes.

Two cubes connected by edges.

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**Induction Step Idea**

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Use recursive definition into two subcubes.

Two cubes connected by edges.

Case 1: Count edges inside subcube inductively.

Case 2: Count inside and across.
Induction Step Idea

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

Use recursive definition into two subcubes.

Two cubes connected by edges.

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Induction Step

**Thm:** For any cut \((S, V – S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).
Induction Step

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step.
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**Proof: Induction Step.**
Recursive definition:
Induction Step

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof: Induction Step.**
Recursive definition:
\[ H_0 = (V_0, E_0), H_1 = (V_1, E_1), \] edges \(E_x\) that connect them.
**Induction Step**

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step.

Recursive definition:

\[
H_0 = (V_0, E_0), H_1 = (V_1, E_1), \text{ edges } E_x \text{ that connect them.} \\
H = (V_0 \cup V_1, E_0 \cup E_1 \cup E_x)
\]
Induction Step

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

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S = S_0 \cup S_1 \text{ where } S_0 \text{ in first, and } S_1 \text{ in other.}
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**Case 1:** \(|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2\)
Thm: For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

Proof: Induction Step.
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Case 1: \(|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2\)
Both \(S_0\) and \(S_1\) are small sides.
**Induction Step**

**Thm:** For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

**Proof: Induction Step.**

Recursive definition:

$H_0 = (V_0, E_0), H_1 = (V_1, E_1)$, edges $E_x$ that connect them.

$H = (V_0 \cup V_1, E_0 \cup E_1 \cup E_x)$

$S = S_0 \cup S_1$ where $S_0$ in first, and $S_1$ in other.

**Case 1:** $|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2$

Both $S_0$ and $S_1$ are small sides. So by induction.
Induction Step

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

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Both \(S_0\) and \(S_1\) are small sides. So by induction.

Edges cut in \(H_0 \geq |S_0|\).
Induction Step

**Thm:** For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

**Proof: Induction Step.**

Recursive definition:

- $H_0 = (V_0, E_0), H_1 = (V_1, E_1)$, edges $E_x$ that connect them.
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**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

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S = S_0 \cup S_1 \text{ where } S_0 \text{ in first, and } S_1 \text{ in other.}
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**Case 1:** \(|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2\)

Both \(S_0\) and \(S_1\) are small sides. So by induction.

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Thm: For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

Proof: Induction Step.
Recursive definition:
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\begin{align*}
H_0 &= (V_0, E_0), H_1 = (V_1, E_1), \text{edges } E_x \text{ that connect them.} \\
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S &= S_0 \cup S_1 \text{ where } S_0 \text{ in first, and } S_1 \text{ in other.}
\end{align*}
\]

Case 1: \(|S_0| \leq \frac{|V_0|}{2}, |S_1| \leq \frac{|V_1|}{2}\)
Both \(S_0\) and \(S_1\) are small sides. So by induction.
- Edges cut in \(H_0 \geq |S_0|\).
- Edges cut in \(H_1 \geq |S_1|\).

Total cut edges \(\geq |S_0| + |S_1| = |S|\).
Induction Step

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof: Induction Step.**

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H_0 = (V_0, E_0), H_1 = (V_1, E_1), \text{ edges } E_x \text{ that connect them.}
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Both \(S_0\) and \(S_1\) are small sides. So by induction.

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Total cut edges \(\geq |S_0| + |S_1| = |S|\).
Thm: For any cut \((S, V-S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

\[|S_0| \geq |V_0|/2.\]
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

\[ |S_0| \geq |V_0|/2. \]

Recall Case 1: \(|S_0|, |S_1| \leq |V|/2\)

\[ |S_1| \leq |V_1|/2 \text{ since } |S| \leq |V|/2. \]
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

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|S_0| \geq |V_0|/2.
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Recall Case 1: \(|S_0|, |S_1| \leq |V|/2

\(|S_1| \leq |V_1|/2 \) since \(|S| \leq |V|/2.

\[\implies \geq |S_1| \text{ edges cut in } E_1.\]
Induction Step. Case 2.

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

\[ |S_0| \geq |V_0|/2. \]
Recall Case 1: \(|S_0|, |S_1| \leq |V|/2\)
\[ |S_1| \leq |V_1|/2 \text{ since } |S| \leq |V|/2. \]
\[ \implies \geq |S_1| \text{ edges cut in } E_1. \]
\[ |S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2 \]

Edges in \(E_x\) connect corresponding nodes.
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

- \(|S_0| \geq |V_0|/2.\)
- Recall Case 1: \(|S_0|, |S_1| \leq |V|/2\)
- \(|S_1| \leq |V_1|/2\) since \(|S| \leq |V|/2.\)
- \(\implies \geq |S_1|\) edges cut in \(E_1.\)
- \(|S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2\)
- \(\implies \geq |V_0| - |S_0|\) edges cut in \(E_0.\)
**Thm:** For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

**Proof:** Induction Step. Case 2.

$|S_0| \geq |V_0|/2$.

Recall Case 1: $|S_0|, |S_1| \leq |V|/2$

$|S_1| \leq |V_1|/2$ since $|S| \leq |V|/2$.

$\implies \geq |S_1|$ edges cut in $E_1$.

$|S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2$

$\implies \geq |V_0| - |S_0|$ edges cut in $E_0$.

Edges in $E_x$ connect corresponding nodes.
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

\[|S_0| \geq |V_0|/2.\]

Recall Case 1: \(|S_0|, |S_1| \leq |V|/2\)

\[|S_1| \leq |V_1|/2 \text{ since } |S| \leq |V|/2.\]

\[\implies \geq |S_1| \text{ edges cut in } E_1.\]

\[|S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2\]

\[\implies \geq |V_0| - |S_0| \text{ edges cut in } E_0.\]

Edges in \(E_x\) connect corresponding nodes.

\[\implies = |S_0| - |S_1| \text{ edges cut in } E_x.\]
Induction Step. Case 2.

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

\[|S_0| \geq |V_0|/2.\]

Recall Case 1: \(|S_0|, |S_1| \leq |V|/2\)

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Edges in \(E_x\) connect corresponding nodes.

\[\implies = |S_0| - |S_1| \text{ edges cut in } E_x.\]
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

\[
|S_0| \geq \frac{|V_0|}{2}.
\]

Recall Case 1: \(|S_0|, |S_1| \leq \frac{|V|}{2}

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|S_1| \leq \frac{|V_1|}{2} \text{ since } |S| \leq \frac{|V|}{2}.
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|S_0| \geq \frac{|V_0|}{2} \implies |V_0 - S| \leq \frac{|V_0|}{2}
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\implies \geq |V_0| - |S_0| \text{ edges cut in } E_0.
\]

Edges in \(E_x\) connect corresponding nodes.

\[
\implies = |S_0| - |S_1| \text{ edges cut in } E_x.
\]

Total edges cut:
Thm: For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).


\[ |S_0| \geq |V_0|/2. \]

Recall Case 1: \(|S_0|, |S_1| \leq |V|/2\)
\[ |S_1| \leq |V_1|/2 \text{ since } |S| \leq |V|/2. \]
\[ \implies \geq |S_1| \text{ edges cut in } E_1. \]
\[ |S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2 \]
\[ \implies \geq |V_0| - |S_0| \text{ edges cut in } E_0. \]

Edges in \(E_x\) connect corresponding nodes.
\[ \implies |S_0| - |S_1| \text{ edges cut in } E_x. \]

Total edges cut:
\[ \geq \]
**Thm:** For any cut \((S, V \setminus S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

\[
|S_0| \geq |V_0|/2. \\
\text{Recall Case 1: } |S_0|, |S_1| \leq |V|/2 \\
|S_1| \leq |V_1|/2 \text{ since } |S| \leq |V|/2. \\
\implies \geq |S_1| \text{ edges cut in } E_1. \\
|S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2 \\
\implies \geq |V_0| - |S_0| \text{ edges cut in } E_0.
\]

Edges in \(E_x\) connect corresponding nodes.

\[
\implies = |S_0| - |S_1| \text{ edges cut in } E_x.
\]

Total edges cut:

\[
\geq |S_1|
\]
**Induction Step. Case 2**.

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

\[
|S_0| \geq |V_0|/2.
\]

Recall Case 1: \(|S_0|, |S_1| \leq |V|/2 \)

\(|S_1| \leq |V_1|/2 \) since \(|S| \leq |V|/2\).

\[
\implies S_1 \text{ edges cut in } E_1.
\]

\[
|S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2
\]

\[
\implies \geq |V_0| - |S_0| \text{ edges cut in } E_0.
\]

Edges in \(E_x\) connect corresponding nodes.

\[
\implies = |S_0| - |S_1| \text{ edges cut in } E_x.
\]

Total edges cut:

\[
\geq |S_1| + |V_0| - |S_0|
\]
Thm: For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).


\[
|S_0| \geq \frac{|V_0|}{2}.
\]

Recall Case 1: \(|S_0|, |S_1| \leq \frac{|V|}{2}\)

\(|S_1| \leq \frac{|V_1|}{2}\) since \(|S| \leq \frac{|V|}{2}\).

\[
\implies \geq |S_1| \text{ edges cut in } E_1.
\]

\(|S_0| \geq \frac{|V_0|}{2} \implies |V_0 - S| \leq \frac{|V_0|}{2}
\]

\[
\implies \geq |V_0| - |S_0| \text{ edges cut in } E_0.
\]

Edges in \(E_x\) connect corresponding nodes.

\[
\implies = |S_0| - |S_1| \text{ edges cut in } E_x.
\]

Total edges cut:

\[
\geq |S_1| + |V_0| - |S_0| + |S_0| - |S_1|
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\[ |S_1| \leq |V_1|/2 \text{ since } |S| \leq |V|/2. \]
\[ \implies \geq |S_1| \text{ edges cut in } E_1. \]
\[ |S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2 \]
\[ \implies \geq |V_0| - |S_0| \text{ edges cut in } E_0. \]

Edges in \(E_x\) connect corresponding nodes.
\[ \implies = |S_0| - |S_1| \text{ edges cut in } E_x. \]

Total edges cut:
\[ \geq |S_1| + |V_0| - |S_0| + |S_0| - |S_1| = |V_0| \]
Induction Step. Case 2.

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof: Induction Step. Case 2.**

\[ |S_0| \geq \frac{|V_0|}{2}. \]

Recall Case 1: \(|S_0|, |S_1| \leq \frac{|V|}{2} \]
\[ |S_1| \leq \frac{|V_1|}{2} \text{ since } |S| \leq \frac{|V|}{2}. \]
\[ \implies \geq |S_1| \text{ edges cut in } E_1. \]
\[ |S_0| \geq \frac{|V_0|}{2} \implies |V_0 - S| \leq \frac{|V_0|}{2} \]
\[ \implies \geq |V_0| - |S_0| \text{ edges cut in } E_0. \]

Edges in \(E_x\) connect corresponding nodes.
\[ \implies = |S_0| - |S_1| \text{ edges cut in } E_x. \]

Total edges cut:
\[ \geq |S_1| + |V_0| - |S_0| + |S_0| - |S_1| = |V_0| \]
\[ \frac{|V_0|}{2} \geq |S| \]
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

Induction Step. Case 2.

Edges in \(E_x\) connect corresponding nodes.

\[|S_0| \geq |V_0|/2.\]

Recall Case 1: 

\[|S_0|, |S_1| \leq |V|/2\]

\[|S_1| \leq |V_1|/2\] since \(|S| \leq |V|/2\).

\[\implies \geq |S_1| \text{ edges cut in } E_1.\]

\[|S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2\]

\[\implies \geq |V_0| - |S_0| \text{ edges cut in } E_0.\]

Total edges cut:

\[\geq |S_1| + |V_0| - |S_0| + |S_0| - |S_1| = |V_0|\]

\[|V_0| = |V|/2 \geq |S|.\]
**Thm:** For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

**Proof:** Induction Step. Case 2.

Recall Case 1: $|S_0|, |S_1| \leq |V|/2$

$|S_1| \leq |V_1|/2$ since $|S| \leq |V|/2$.

$\implies |S_1|$ edges cut in $E_1$.

$|S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2$

$\implies \geq |V_0| - |S_0|$ edges cut in $E_0$.

Edges in $E_x$ connect corresponding nodes.

$\implies = |S_0| - |S_1|$ edges cut in $E_x$.

Total edges cut:

$\geq |S_1| + |V_0| - |S_0| + |S_0| - |S_1| = |V_0|$

$|V_0| = |V|/2 \geq |S|$.
Thm: For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).


\[ |S_0| \geq \frac{|V_0|}{2}. \]
Recall Case 1: \(|S_0|, |S_1| \leq \frac{|V|}{2}
\]
\[ |S_1| \leq \frac{|V_1|}{2} \text{ since } |S| \leq \frac{|V|}{2}. \]
\[ \implies \geq |S_1| \text{ edges cut in } E_1. \]
\[ |S_0| \geq \frac{|V_0|}{2} \implies |V_0 - S| \leq \frac{|V_0|}{2} \]
\[ \implies \geq |V_0| - |S_0| \text{ edges cut in } E_0. \]

Edges in \(E_x\) connect corresponding nodes.
\[ \implies = |S_0| - |S_1| \text{ edges cut in } E_x. \]

Total edges cut:
\[ \geq |S_1| + |V_0 - S_0| + |S_0 - S_1| = |V_0| \]
\[ |V_0| = \frac{|V|}{2} \geq |S|. \]

Also, case 3 where \(|S_1| \geq \frac{|V|}{2}\) is symmetric.
The cuts in the hypercubes are exactly the transitions from 0 sets to 1 set on boolean functions on $\{0, 1\}^n$. 
The cuts in the hypercubes are exactly the transitions from 0 sets to 1 set on boolean functions on \( \{0, 1\}^n \).

Central area of study in computer science!
Hypercubes and Boolean Functions.

The cuts in the hypercubes are exactly the transitions from 0 sets to 1 set on boolean functions on $\{0, 1\}^n$.

Central area of study in computer science!

Yes/No Computer Programs $\equiv$ Boolean function on $\{0, 1\}^n$
The cuts in the hypercubes are exactly the transitions from 0 sets to 1 set on boolean functions on $\{0, 1\}^n$.

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Central object of study.
The cuts in the hypercubes are exactly the transitions from 0 sets to 1 set on boolean functions on $\{0, 1\}^n$. 
Hypercubes and Boolean Functions.

The cuts in the hypercubes are exactly the transitions from 0 sets to 1 set on boolean functions on \( \{0, 1\}^n \).

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Central area of study in computer science!

Yes/No Computer Programs \( \equiv \) Boolean function on \( \{0, 1\}^n \)
The cuts in the hypercubes are exactly the transitions from 0 sets to 1 set on boolean functions on $\{0, 1\}^n$.

Central area of study in computer science!

Yes/No Computer Programs $\equiv$ Boolean function on $\{0, 1\}^n$

Central object of study.
Modular Arithmetic.

Applications: cryptography, error correction.
Key idea for modular arithmetic.

Theorem: If \( d \mid x \) and \( d \mid y \), then \( d \mid (y - x) \).
Key idea for modular arithmetic.

Theorem: If $d|x$ and $d|y$, then $d|(y - x)$.

Proof:
Theorem: If \( d|x \) and \( d|y \), then \( d|(y - x) \).

Proof:
\[
x = ad, \quad y = bd,
\]
Key idea for modular arithmetic.

Theorem: If $d|x$ and $d|y$, then $d|(y - x)$.

Proof:

$x = ad, y = bd,\hspace{1cm} (x - y) = (ad - bd) = d(a - b) \implies d|(x - y)$. 

$_{\Box}$
Key idea for modular arithmetic.

Theorem: If $d|x$ and $d|y$, then $d|(y - x)$.

Proof:
$x = ad$, $y = bd$,
$(x - y) = (ad - bd) = d(a - b) \implies d|(x - y)$.

Theorem: Every number $n \geq 2$ can be represented as a product of primes.
Key idea for modular arithmetic.

Theorem: If $d|x$ and $d|y$, then $d|(y - x)$.

Proof:

$x = ad$, $y = bd$,

$$(x - y) = (ad - bd) = d(a - b) \implies d|(x - y).$$

Theorem: Every number $n \geq 2$ can be represented as a product of primes.

Proof: Either prime, or $n = a \times b$, and use strong induction. (Uniqueness? Later.)
Next Up.

Modular Arithmetic.
If it is 1:00 now.
Clock Math

If it is 1:00 now.
What time is it in 2 hours?

3:00!

What time is it in 5 hours?
6:00!

What time is it in 15 hours?
16:00!

Actually 4:00.
16 is the "same as 4" with respect to a 12 hour clock system.
Clock time equivalent up to addition/subtraction of 12.

What time is it in 100 hours?
101:00!
or 5:00.
101 = 12 × 8 + 5.
5 is the same as 101 for a 12 hour clock system.
Clock time equivalent up to addition of any integer multiple of 12.
Custom is only to use the representative in \{12,1,...,11\} (Almost remainder, except for 12 and 0 are equivalent.)

15 / 30
Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!

15 / 30
If it is 1:00 now.
  What time is it in 2 hours? 3:00!
  What time is it in 5 hours?
Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
Clock Math

If it is 1:00 now.
  What time is it in 2 hours? 3:00!
  What time is it in 5 hours? 6:00!
  What time is it in 15 hours?

15 / 30
Clock Math

If it is 1:00 now.
  What time is it in 2 hours? 3:00!
  What time is it in 5 hours? 6:00!
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Clock Math

If it is 1:00 now.
  What time is it in 2 hours?  3:00!
  What time is it in 5 hours?  6:00!
  What time is it in 15 hours?  16:00!
    Actually 4:00.
If it is 1:00 now.
  What time is it in 2 hours? 3:00!
  What time is it in 5 hours? 6:00!
  What time is it in 15 hours? 16:00!
    Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.
Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
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16 is the “same as 4” with respect to a 12 hour clock system.
Clock time equivalent up to to addition/subtraction of 12.
Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
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Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!
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16 is the “same as 4” with respect to a 12 hour clock system. Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours?
Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
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    Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.
Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00!
Clock Math

If it is 1:00 now.
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  What time is it in 15 hours? 16:00!
    Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.
Clock time equivalent up to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.
If it is 1:00 now.
What time is it in 2 hours? 3:00!
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What time is it in 100 hours? 101:00! or 5:00.
$101 = 12 \times 8 + 5$. 
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If it is 1:00 now.
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What time is it in 100 hours? 101:00! or 5:00.
\[ 101 = 12 \times 8 + 5. \]
5 is the same as 101 for a 12 hour clock system.
Clock Math

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Clock Math

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Clock Math

If it is 1:00 now.
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What time is it in 100 hours? 101:00! or 5:00.

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5 is the same as 101 for a 12 hour clock system.
Clock time equivalent up to addition of any integer multiple of 12.

Custom is only to use the representative in \{12, 1, \ldots, 11\}
(Almost remainder, except for 12 and 0 are equivalent.)
Day of the week.

Today is Thursday.
Day of the week.

Today is Thursday.
What day is it a year from now?
Day of the week.

Today is Thursday.

What day is it a year from now? on September 17, 2021?
Day of the week.

Today is Thursday.
What day is it a year from now? on September 17, 2021?
Number days.
Today is Thursday.

What day is it a year from now? on September 17, 2021?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.
Day of the week.

Today is Thursday.

What day is it a year from now? on September 17, 2021?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.
Day of the week.

Today is Thursday.

What day is it a year from now? on September 17, 2021?
Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
Today is Thursday.

What day is it a year from now? on September 17, 2021?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

5 days from now.

11 days from now is day 1 which is Monday!
Day of the week.

Today is Thursday.
What day is it a year from now? on September 17, 2021?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
5 days from now. day 9
Day of the week.

Today is Thursday.
What day is it a year from now? on September 17, 2021?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
5 days from now. day 9 or day 2
Day of the week.

Today is Thursday.

What day is it a year from now? on September 17, 2021?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

5 days from now. day 9 or day 2 or Tuesday.

Next year is not a leap year.

So 365 days from now.

Day 4+365 or day 369.

Smallest representation: subtract 7 until smaller than 7.

divide and get remainder.

369/7 leaves quotient of 52 and remainder 3.

369 = 7*(52) + 5 or September 18, 2020 is a Friday.
Today is Thursday.

What day is it a year from now? on September 17, 2021?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
5 days from now. day 9 or day 2 or Tuesday.
25 days from now.
Day of the week.

Today is Thursday.
What day is it a year from now? on September 17, 2021?
Number days.
  0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
  5 days from now. day 9 or day 2 or Tuesday.
  25 days from now. day 29
Day of the week.

Today is Thursday.
What day is it a year from now? on September 17, 2021?
Number days.
  0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
  5 days from now. day 9 or day 2 or Tuesday.
  25 days from now. day 29 or day 1.
Day of the week.

Today is Thursday.

What day is it a year from now? on September 17, 2021?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

5 days from now. day 9 or day 2 or Tuesday.

25 days from now. day 29 or day 1. 29 = (7)4 + 1
Day of the week.

Today is Thursday.
What day is it a year from now? on September 17, 2021?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
5 days from now. day 9 or day 2 or Tuesday.
25 days from now. day 29 or day 1. \(29 = (7)4 + 1\)
two days are equivalent up to addition/subtraction of multiple of 7.
Day of the week.

Today is Thursday.

What day is it a year from now? on September 17, 2021?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

5 days from now. day 9 or day 2 or Tuesday.

25 days from now. day 29 or day 1. $29 = (7)4 + 1$

two days are equivalent up to addition/subtraction of multiple of 7.

11 days from now
Day of the week.

Today is Thursday.

What day is it a year from now? on September 17, 2021?
Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

5 days from now. day 9 or day 2 or Tuesday.
25 days from now. day 29 or day 1. 29 = (7)4 + 1
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from now is day 1

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Day of the week.

Today is Thursday.

What day is it a year from now? on September 17, 2021?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

5 days from now. day 9 or day 2 or Tuesday.

25 days from now. day 29 or day 1. $29 = (7)4 + 1$

two days are equivalent up to addition/subtraction of multiple of 7.

11 days from now is day 1 which is Monday!
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Day of the week.

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What day is it a year from now?
  Next year is not a leap year. So 365 days from now.
Day of the week.

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What day is it a year from now?
Next year is not a leap year. So 365 days from now.
Day 4+365 or day 369.
Day of the week.

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Smallest representation:
Today is Thursday.
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  two days are equivalent up to addition/subtraction of multiple of 7.
  11 days from now is day 1 which is Monday!

What day is it a year from now?
  Next year is not a leap year. So 365 days from now.
  Day 4+365 or day 369.
Smallest representation:
  subtract 7 until smaller than 7.
Today is Thursday.
What day is it a year from now? on September 17, 2021?
Number days.
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Day 4+365 or day 369.
Smallest representation:
subtract 7 until smaller than 7.
divide and get remainder.
Day of the week.

Today is Thursday.
  What day is it a year from now? on September 17, 2021?
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  0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

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  5 days from now. day 9 or day 2 or Tuesday.
  25 days from now. day 29 or day 1. \[29 = (7)4 + 1\]
    two days are equivalent up to addition/subtraction of multiple of 7.
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  Day 4+365 or day 369.
Smallest representation:
  subtract 7 until smaller than 7.
  divide and get remainder.
  369/7
Day of the week.

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11 days from now is day 1 which is Monday!

What day is it a year from now?
Next year is not a leap year. So 365 days from now.
Day 4+365 or day 369.
Smallest representation:
subtract 7 until smaller than 7.
divide and get remainder.
369/7 leaves quotient of 52 and remainder 3.
Day of the week.

Today is Thursday.

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Day 4+365 or day 369.

Smallest representation:
subtract 7 until smaller than 7.
divide and get remainder.

369/7 leaves quotient of 52 and remainder 3. $369 = 7(52) + 5$
Day of the week.

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Day of the week.

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or September 18, 2020 is a Friday.
Years and years...

80 years from now?

20 leap years.

20 \times 366 \times 20 \text{ days}

60 regular years.

365 \times 60 \text{ days}

Today is day 4.

Remainder of 366 when dividing by 7:

52 \times 7 + 2.

Remainder of 365 when dividing by 7:

1.

Today is day 4.

Get Day: 4 + 2 \times 20 + 1 \times 60 = 104

Remainder when dividing by 7:

104 = 14 \times 7 + 6.

Or September 18, 2100 is Saturday.

Further Simplify Calculation:

20 has remainder 6 when divided by 7.

60 has remainder 4 when divided by 7.

Get Day: 2 + 2 \times 6 + 1 \times 4 = 18

Or Day 6.

September 18, 2100 is Saturday.

"Reduce" at any time in calculation!
Years and years...

80 years from now? 20 leap years.
Years and years...

80 years from now? 20 leap years. $366 \times 20$ days
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Years and years...

80 years from now? 20 leap years. $366 \times 20$ days
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Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. 

Equivalent to?
Hmm.
What is remainder of 366 when dividing by 7?
52
$7 \times 52 + 2$.
What is remainder of 365 when dividing by 7?
1
Today is day 4.
Get Day: 4 + 2 × 20 + 1 × 60.
Remainder when dividing by 7?
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17 / 30
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“Reduce” at any time in calculation!
Modular Arithmetic: refresher.

\[x \text{ is congruent to } y \text{ modulo } m\] or “\(x \equiv y \pmod{m}\)” if and only if \((x - y)\) is divisible by \(m\).
Modular Arithmetic: refresher.

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...or \( x = y + km \) for some integer \( k \).
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Mod 7 equivalence classes:
Modular Arithmetic: refresher.

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...or \(x = y + km\) for some integer \(k\).

Mod 7 equivalence classes:
\[
\{\ldots, -7, 0, 7, 14, \ldots\}
\]
Modular Arithmetic: refresher.

*x is congruent to y modulo m* or “*x ≡ y (mod m)*” if and only if *(x – y)* is divisible by *m.*

...or *x* and *y* have the same remainder w.r.t. *m.*

...or *x = y + km* for some integer *k.*

Mod 7 equivalence classes:

{..., –7, 0, 7, 14, ...}  {..., –6, 1, 8, 15, ...}
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Modular Arithmetic: refresher.

$x$ is congruent to $y$ modulo $m$ or “$x \equiv y \pmod{m}$” if and only if $(x - y)$ is divisible by $m$.

...or $x$ and $y$ have the same remainder w.r.t. $m$.

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Mod 7 equivalence classes:

\{…,−7,0,7,14,…\} \quad \{…,−6,1,8,15,…\} ...

**Useful Fact:** Addition, subtraction, multiplication can be done with any equivalent $x$ and $y$. 
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\[ \{ \ldots, -7, 0, 7, 14, \ldots \} \quad \{ \ldots, -6, 1, 8, 15, \ldots \} \quad \ldots \]

**Useful Fact:** Addition, subtraction, multiplication can be done with any equivalent \( x \) and \( y \).

or “\( a \equiv c \pmod{m} \) and \( b \equiv d \pmod{m} \)”
Modular Arithmetic: refresher.

\( x \) is congruent to \( y \) modulo \( m \) or “\( x \equiv y \pmod{m} \)” if and only if \( (x - y) \) is divisible by \( m \).

...or \( x \) and \( y \) have the same remainder w.r.t. \( m \).

...or \( x = y + km \) for some integer \( k \).

Mod 7 equivalence classes:
\{\ldots, -7, 0, 7, 14, \ldots\}  \quad \{\ldots, -6, 1, 8, 15, \ldots\} \ldots

**Useful Fact:** Addition, subtraction, multiplication can be done with any equivalent \( x \) and \( y \).

or “\( a \equiv c \pmod{m} \) and \( b \equiv d \pmod{m} \)
\( \implies a + b \equiv c + d \pmod{m} \) and \( a \cdot b \equiv c \cdot d \pmod{m} \)”
Modular Arithmetic: refresher.

\( x \) is congruent to \( y \) modulo \( m \) or “\( x \equiv y \ (\text{mod} \ m) \)” if and only if \( (x - y) \) is divisible by \( m \).

...or \( x \) and \( y \) have the same remainder w.r.t. \( m \).

...or \( x = y + km \) for some integer \( k \).

Mod 7 equivalence classes:
\[
\{..., -7, 0, 7, 14, ...\} \quad \{..., -6, 1, 8, 15, ...\} \quad ...
\]

**Useful Fact:** Addition, subtraction, multiplication can be done with any equivalent \( x \) and \( y \).

or “\( a \equiv c \ (\text{mod} \ m) \) and \( b \equiv d \ (\text{mod} \ m) \)
\[\Rightarrow a + b \equiv c + d \ (\text{mod} \ m) \) and \( a \cdot b \equiv c \cdot d \ (\text{mod} \ m) \)”

**Proof:** If \( a \equiv c \ (\text{mod} \ m) \), then \( a = c + km \) for some integer \( k \).
Modular Arithmetic: refresher.

\( x \) is congruent to \( y \) modulo \( m \) or “\( x \equiv y \pmod{m} \)” if and only if \( (x - y) \) is divisible by \( m \).

...or \( x \) and \( y \) have the same remainder w.r.t. \( m \).

...or \( x = y + km \) for some integer \( k \).

Mod 7 equivalence classes:
\{\ldots, -7, 0, 7, 14, \ldots\} \quad \{\ldots, -6, 1, 8, 15, \ldots\} \ldots

Useful Fact: Addition, subtraction, multiplication can be done with any equivalent \( x \) and \( y \).

or “\( a \equiv c \pmod{m} \) and \( b \equiv d \pmod{m} \)
\implies a + b \equiv c + d \pmod{m} \) and \( a \cdot b \equiv c \cdot d \pmod{m} \)”

Proof: If \( a \equiv c \pmod{m} \), then \( a = c + km \) for some integer \( k \).
If \( b \equiv d \pmod{m} \), then \( b = d + jm \) for some integer \( j \).
Modular Arithmetic: refresher.

\[ x \text{ is congruent to } y \text{ modulo } m \] or “\( x \equiv y \pmod{m} \)” if and only if \( (x - y) \) is divisible by \( m \).

...or \( x \) and \( y \) have the same remainder w.r.t. \( m \).

...or \( x = y + km \) for some integer \( k \).

Mod 7 equivalence classes:
\[
\{\ldots, -7, 0, 7, 14, \ldots\} \quad \{\ldots, -6, 1, 8, 15, \ldots\} \quad \ldots
\]

**Useful Fact:** Addition, subtraction, multiplication can be done with any equivalent \( x \) and \( y \).

or “\( a \equiv c \pmod{m} \) and \( b \equiv d \pmod{m} \)\n
\[ \implies a + b \equiv c + d \pmod{m} \text{ and } a \cdot b \equiv c \cdot d \pmod{m} \]”

**Proof:** If \( a \equiv c \pmod{m} \), then \( a = c + km \) for some integer \( k \).

If \( b \equiv d \pmod{m} \), then \( b = d + jm \) for some integer \( j \).

Therefore,
Modular Arithmetic: refresher.

**x is congruent to y modulo m** or “$x \equiv y \pmod{m}$” if and only if $(x - y)$ is divisible by $m$.

...or $x$ and $y$ have the same remainder w.r.t. $m$.

...or $x = y + km$ for some integer $k$.

Mod 7 equivalence classes:
{...,$-7,0,7,14,...$} {...,$-6,1,8,15,...$} ...

**Useful Fact:** Addition, subtraction, multiplication can be done with any equivalent $x$ and $y$.

or “$a \equiv c \pmod{m}$ and $b \equiv d \pmod{m}$

$\implies a + b \equiv c + d \pmod{m}$ and $a \cdot b \equiv c \cdot d \pmod{m}$”

**Proof:** If $a \equiv c \pmod{m}$, then $a = c + km$ for some integer $k$.

If $b \equiv d \pmod{m}$, then $b = d + jm$ for some integer $j$.

Therefore, $a + b = c + d + (k+j)m$
Modular Arithmetic: refresher.

*x is congruent to* $y$ *modulo* $m$ *or “* $x \equiv y \pmod{m}$ “*

if and only if $(x - y)$ is divisible by $m$.

...or $x$ and $y$ have the same remainder w.r.t. $m$.

...or $x = y + km$ for some integer $k$.

Mod 7 equivalence classes:

$\{\ldots, -7, 0, 7, 14, \ldots\}$  $\{\ldots, -6, 1, 8, 15, \ldots\}$ ...

**Useful Fact:** Addition, subtraction, multiplication can be done with any equivalent $x$ and $y$.

or “ $a \equiv c \pmod{m}$ and $b \equiv d \pmod{m}$

$\implies a + b \equiv c + d \pmod{m}$ and $a \cdot b \equiv c \cdot d \pmod{m}$”

**Proof:** If $a \equiv c \pmod{m}$, then $a = c + km$ for some integer $k$.

If $b \equiv d \pmod{m}$, then $b = d + jm$ for some integer $j$.

Therefore, $a + b = c + d + (k + j)m$ and since $k + j$ is integer.
Modular Arithmetic: refresher.

\( x \) is congruent to \( y \) modulo \( m \) or “\( x \equiv y \pmod{m} \)” if and only if \( (x - y) \) is divisible by \( m \).

...or \( x \) and \( y \) have the same remainder w.r.t. \( m \).

...or \( x = y + km \) for some integer \( k \).

Mod 7 equivalence classes:
\[ \{ \ldots, -7, 0, 7, 14, \ldots \} \quad \{ \ldots, -6, 1, 8, 15, \ldots \} \ldots \]

**Useful Fact:** Addition, subtraction, multiplication can be done with any equivalent \( x \) and \( y \).

or “\( a \equiv c \pmod{m} \) and \( b \equiv d \pmod{m} \)
\[ \implies a + b \equiv c + d \pmod{m} \) and \( a \cdot b \equiv c \cdot d \pmod{m} \)”

**Proof:** If \( a \equiv c \pmod{m} \), then \( a = c + km \) for some integer \( k \). If \( b \equiv d \pmod{m} \), then \( b = d + jm \) for some integer \( j \).
Therefore, \( a + b = c + d + (k + j)m \) and since \( k + j \) is integer.
\[ \implies a + b \equiv c + d \pmod{m} \).
Modular Arithmetic: refresher.

*x is congruent to y modulo m* or “*x ≡ y (mod m)*”
if and only if (x – y) is divisible by m.
...or x and y have the same remainder w.r.t. m.
...or *x = y + km* for some integer k.

Mod 7 equivalence classes:
{..., −7, 0, 7, 14, ...}  {..., −6, 1, 8, 15, ...} ...

**Useful Fact:** Addition, subtraction, multiplication can be done with any equivalent x and y.

or “*a ≡ c (mod m) and b ≡ d (mod m)*

\[ \implies a + b ≡ c + d \pmod{m} \] and \[ a \cdot b ≡ c \cdot d \pmod{m} \]”

**Proof:** If *a ≡ c (mod m)*, then *a = c + km* for some integer *k*.
If *b ≡ d (mod m)*, then *b = d + jm* for some integer *j*.
Therefore, \[ a + b = c + d + (k + j)m \] and since \( k + j \) is integer.
\[ \implies a + b ≡ c + d \pmod{m}. \]
Modular Arithmetic: refresher.

\( x \) is congruent to \( y \) modulo \( m \) or \( "x \equiv y \pmod{m}\)"
if and only if \( (x - y) \) is divisible by \( m \).
...or \( x \) and \( y \) have the same remainder w.r.t. \( m \).
...or \( x = y + km \) for some integer \( k \).

Mod 7 equivalence classes:
\{ \ldots, -7, 0, 7, 14, \ldots \} \quad \{ \ldots, -6, 1, 8, 15, \ldots \} \ldots

**Useful Fact:** Addition, subtraction, multiplication can be done with any equivalent \( x \) and \( y \).

or \( "a \equiv c \pmod{m} \) and \( b \equiv d \pmod{m}\)

\[ \Rightarrow a + b \equiv c + d \pmod{m} \) and \( a \cdot b \equiv c \cdot d \pmod{m}" \]

**Proof:** If \( a \equiv c \pmod{m} \), then \( a = c + km \) for some integer \( k \).
If \( b \equiv d \pmod{m} \), then \( b = d + jm \) for some integer \( j \).
Therefore, \( a + b = c + d + (k + j)m \) and since \( k + j \) is integer.
\[ \Rightarrow a + b \equiv c + d \pmod{m} \].

Can calculate with representative in \( \{ 0, \ldots, m-1 \} \).
Notation

\[ x \pmod{m} \text{ or } \text{mod}(x, m) \]
Notation

\[ x \equiv (\text{mod } m) \text{ or } \text{mod } (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).
Notation

\[ x \pmod{m} \text{ or } \mod(x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).
Notation

\[ x \pmod{m} \text{ or } \text{mod} (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[
\text{mod} (x, m) = x - \lfloor \frac{x}{m} \rfloor m
\]
Notation

\[ x \pmod{m} \text{ or } \mod(x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m - 1\} \).

\[
\mod(x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]
\[\left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.}\]
Notation

$x \pmod{m}$ or $\text{mod}(x,m)$

- remainder of $x$ divided by $m$ in $\{0, \ldots, m-1\}$.

\[
\text{mod}(x,m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]

$\left\lfloor \frac{x}{m} \right\rfloor$ is quotient.

$\text{mod}(29,12) = 29 - \left(\left\lfloor \frac{29}{12} \right\rfloor \right) \times 12$
Notation

\[ x \pmod{m} \] or \[ \text{mod} (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m - 1\} \).

\[ \text{mod} (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]
\[ \left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.} \]

\[ \text{mod} (29, 12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 \]
Notation

$x \pmod{m}$ or $\text{mod} \ (x, m)$

- remainder of $x$ divided by $m$ in $\{0, \ldots, m-1\}$.

$$\text{mod} \ (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m$$

$\left\lfloor \frac{x}{m} \right\rfloor$ is quotient.

$$\text{mod} \ (29, 12) = 29 - (\left\lfloor \frac{29}{12} \right\rfloor) \times 12 = 29 - (2) \times 12 = 4$$
Notation

\( x \pmod{m} \) or \( \text{mod} (x, m) \)

- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[
\text{mod} (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]

\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[
\text{mod} (29, 12) = 29 - \left(\left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = 5
\]
Notation

\[ x \pmod{m} \text{ or } \mod(x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m - 1\} \).

\[ \mod(x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]
\[ \left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.} \]

\[ \mod(29, 12) = 29 - \left(\left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = 4 \]

Work in this system.
Notation

\[ x \ (\text{mod} \ m) \text{ or } \mod (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[ \mod (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]
\[ \left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.} \]

\[ \mod (29, 12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 \]
\[ = 4 \]

Work in this system.
\[ a \equiv b \ (\text{mod} \ m). \]
Notation

\[ x \pmod{m} \text{ or } \text{mod} (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[ \text{mod} (x, m) = x - \lfloor \frac{x}{m} \rfloor m \]
\[ \lfloor \frac{x}{m} \rfloor \text{ is quotient.} \]

\[
\text{mod} (29, 12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12 = 29 - (2) \times 12 = \mathbf{4} = 5
\]

Work in this system.

\[ a \equiv b \pmod{m}. \]
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).
Notation

\[ x \pmod{m} \text{ or } \mod(x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m - 1\} \).

\[ \mod(x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]
\[ \left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.} \]

\[ \mod(29, 12) = 29 - (\left\lfloor \frac{29}{12} \right\rfloor) \times 12 = 29 - (2) \times 12 = 4 \]

Work in this system.

\[ a \equiv b \pmod{m}. \]
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)
Notation

\( x \pmod{m} \) or \( \text{mod} (x, m) \)
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[ \text{mod} (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]

\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[ \text{mod} (29, 12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = 4 = 5 \]

Work in this system.

\( a \equiv b \pmod{m} \).
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\( 6 \equiv \)
Notation

\[ x \pmod{m} \text{ or } \text{mod}(x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m - 1\} \).

\[ \text{mod}(x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]
\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[ \text{mod}(29, 12) = 29 - \left(\left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = 4 \]

Work in this system.
\( a \equiv b \pmod{m} \).
 Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\( 6 \equiv 3 + 3 \)
Notation

\( x \pmod{m} \) or \( \text{mod} (x, m) \)
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[ \text{mod} (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]
\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[ \text{mod} (29, 12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = 5 \]

Work in this system.

\[ a \equiv b \pmod{m} \]
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\( 6 \equiv 3 + 3 \equiv 3 + 10 \)
Notation

\[ x \pmod{m} \text{ or } \mod(x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[
\mod(x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]
\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[
\mod(29, 12) = 29 - (\left\lfloor \frac{29}{12} \right\rfloor) \times 12 = 29 - (2) \times 12 = \text{X} = 5
\]

Work in this system.
\[ a \equiv b \pmod{m}. \]
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\[ 6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}. \]
### Notation

\[ x \pmod{m} \text{ or } \text{mod} \ (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m - 1\} \).

\[ \text{mod} \ (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]

\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[ \text{mod} \ (29, 12) = 29 - (\left\lfloor \frac{29}{12} \right\rfloor) \times 12 = 29 - (2) \times 12 = 4 \]

Work in this system.

\[ a \equiv b \pmod{m}. \]
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\[ 6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}. \]

\[ 6 = \]

Notation

\[ x \pmod{m} \text{ or } \mod(x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[
\mod(x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]
\[ \left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.} \]

\[
\mod(29, 12) = 29 - \left(\left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = 4
\]

Work in this system.
\[ a \equiv b \pmod{m}. \]
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\[ 6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7} . \]

\[ 6 = 3 + 3 \]
Notation

$x \ (\text{mod } m)$ or $\mod (x, m)$
- remainder of $x$ divided by $m$ in $\{0, \ldots, m-1\}$.

$\mod (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m$

$\left\lfloor \frac{x}{m} \right\rfloor$ is quotient.

$\mod (29, 12) = 29 - (\left\lfloor \frac{29}{12} \right\rfloor) \times 12 = 29 - (2 \times 12) = \text{X} = 5$

Work in this system.

$a \equiv b \ (\text{mod } m)$.

Says two integers $a$ and $b$ are equivalent modulo $m$.

**Modulus** is $m$

$6 \equiv 3 + 3 \equiv 3 + 10 \ (\text{mod } 7)$.

$6 = 3 + 3 = 3 + 10$
Notation

\[ x \pmod{m} \text{ or } \; \mod (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[ \mod (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]
\[ \left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.} \]

\[ \mod (29, 12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = \text{X} = 5 \]

Work in this system.
\[ a \equiv b \pmod{m} \]
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\[ 6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7} \]
\[ 6 = 3 + 3 = 3 + 10 \pmod{7} \]
**Notation**

\[ x \pmod{m} \text{ or } \mod(x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[
\mod(x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]
\[ \left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.} \]

\[
\mod(29, 12) = 29 - \left\lfloor \frac{29}{12} \right\rfloor \times 12 = 29 - (2) \times 12 = 4 \]

Work in this system.

\[ a \equiv b \pmod{m}. \]
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\[
6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}.
\]
\[
6 = 3 + 3 = 3 + 10 \pmod{7}.
\]
Generally, not \( 6 \pmod{7} = 13 \pmod{7} \).
Notation

\[ x \pmod{m} \text{ or } \text{mod}(x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[
\text{mod}(x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]
\[ \left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.} \]

\[
\text{mod}(29, 12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = 5
\]

Work in this system.

\[ a \equiv b \pmod{m}. \]
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\[ 6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}. \]

\[ 6 = 3 + 3 = 3 + 10 \pmod{7}. \]

Generally, not \( 6 \pmod{7} = 13 \pmod{7} \).
But probably won’t take off points,
Notation

\( x \pmod{m} \) or \( \text{mod} \ (x, m) \)
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m - 1\} \).

\[
\text{mod} \ (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]

\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[
\text{mod} \ (29, 12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = 4 = 5
\]

Work in this system.

\( a \equiv b \pmod{m} \).
Say two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\( 6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7} \).

\( 6 = 3 + 3 = 3 + 10 \pmod{7} \).

Generally, not \( 6 \pmod{7} = 13 \pmod{7} \).

But probably won’t take off points, still hard for us to read.
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]

**Multiplicative inverse of** \( x \) **is** \( y \) where \( xy = 1 \);
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \iff \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \iff x = \frac{3}{2}. \]

**Multiplicative inverse of** \( x \) **is** \( y \) **where** \( xy = 1; \)

\( 1 \) **is multiplicative identity element.**
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left(\frac{1}{2}\right) \cdot 2x = \left(\frac{1}{2}\right) \cdot 3 \implies x = \frac{3}{2}. \]

**Multiplicative inverse of** \(x\) is \(y\) where \(xy = 1\); \(1\) is multiplicative identity element.

In modular arithmetic, \(1\) is the multiplicative identity element.
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]

**Multiplicative inverse of** \( x \) **is** \( y \) **where** \( xy = 1 \);

1 is multiplicative identity element.

In modular arithmetic, 1 is the multiplicative identity element.

**Multiplicative inverse of** \( x \mod m \) **is** \( y \) **with** \( xy = 1 \mod m \).
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]

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For 4 modulo 7 inverse is 2: \( 2 \cdot 4 \equiv 8 \equiv 1 \mod 7 \).
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Can solve \( 4x = 5 \) (mod 7).
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For 4 modulo 7 inverse is 2: \[ 2 \cdot 4 \equiv 8 \equiv 1 \mod 7. \]

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\[ 2 \cdot 4x = 2 \cdot 5 \mod 7 \]
\[ 8x = 10 \mod 7 \]
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\[
\begin{align*}
2 \cdot 4x & = 2 \cdot 5 \text{ (mod 7)} \\
8x & = 10 \text{ (mod 7)} \\
x & = 3 \text{ (mod 7)}
\end{align*}
\]
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\[
2 \cdot 4x = 2 \cdot 5 \pmod{7} \\
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x = 3 \pmod{7}
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Check!
Inverses and Factors.

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\[ 2 \cdot 4x = 2 \cdot 5 \pmod{7} \]
\[ 8x = 10 \pmod{7} \]
\[ x = 3 \pmod{7} \]

Check! \( 4(3) = 12 = 5 \pmod{7} \).
Inverses and Factors.

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For \( 4 \) modulo \( 7 \) inverse is \( 2 \): \( 2 \cdot 4 \equiv 8 \equiv 1 \) (mod \( 7 \)).

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\[ 2 \cdot 4 \equiv 8 \equiv 1 \pmod{7}. \]

Can solve 4\( x = 5 \) (mod 7).

\[ x = 3 \pmod{7} \implies \text{Check! } 4(3) = 12 = 5 \pmod{7}. \]

For 8 modulo 12: no multiplicative inverse!
Inverses and Factors.

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“Common factor of 4”
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\[ 8k - 12\ell \text{ is a multiple of four for any } \ell \text{ and } k \implies \]
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\( 8k - 12\ell \) is a multiple of four for any \( \ell \) and \( k \) \( \implies \)
\[ 8k \not\equiv 1 \mod 12 \] for any \( k \).
Greatest Common Divisor and Inverses.

**Thm:**
If greatest common divisor of $x$ and $m$, $\gcd(x, m)$, is 1, then $x$ has a multiplicative inverse modulo $m$. 

Proof:

Claim: The set $S = \{0x, 1x, \ldots, (m-1)x\}$ contains $y \equiv 1 \pmod{m}$ if all distinct modulo $m$.

Each of $m$ numbers in $S$ correspond to different one of $m$ equivalence classes modulo $m$.

$\Rightarrow$ One must correspond to 1 modulo $m$.

Inverse Exists!

Proof of Claim:
If not distinct, then $\exists a, b \in \{0, \ldots, m-1\}$, $a \neq b$, where $(ax \equiv bx \pmod{m}) \Rightarrow (a - b)x \equiv 0 \pmod{m}$

Or $(a - b)x = km$ for some integer $k$.

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\( \implies \) \( (a − b) \) factorization contains all primes in \( m \)’s factorization.
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If greatest common divisor of $x$ and $m$, $\gcd(x, m)$, is 1, then $x$ has a multiplicative inverse modulo $m$.

**Proof $\Rightarrow$**:
**Claim:** The set $S = \{0x, 1x, \ldots, (m - 1)x\}$ contains $y \equiv 1 \mod m$ if all distinct modulo $m$.

Each of $m$ numbers in $S$ correspond to different one of $m$ equivalence classes modulo $m$.

$\Rightarrow$ One must correspond to 1 modulo $m$. Inverse Exists!

Proof of Claim: If not distinct, then $\exists a, b \in \{0, \ldots, m - 1\}$, $a \neq b$, where $(ax \equiv bx \mod m) \Rightarrow (a - b)x \equiv 0 \mod m$

Or $(a - b)x = km$ for some integer $k$.

$\gcd(x, m) = 1$
$\Rightarrow$ Prime factorization of $m$ and $x$ do not contain common primes.

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So $(a - b)$ has to be multiple of $m$. 


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So \( (a - b) \) has to be multiple of \( m \).
\( \implies (a - b) \geq m \). But \( a, b \in \{0, \ldots m - 1\} \). Contradiction.
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$\implies (a - b) \geq m$. But $a, b \in \{0, \ldots m - 1\}$. Contradiction.
Proof review. Consequence.

**Thm:** If $\gcd(x, m) = 1$, then $x$ has a multiplicative inverse modulo $m$. 

Proof Sketch:
The set $S = \{0 \cdot x, 1 \cdot x, \ldots, (m - 1) \cdot x\}$ contains $y \equiv 1 \pmod{m}$ if all distinct modulo $m$.

For $x = 4$ and $m = 6$.

$S = \{0 \cdot 4, 1 \cdot 4, 2 \cdot 4, 3 \cdot 4, 4 \cdot 4, 5 \cdot 4\}$

$= \{0, 4, 8, 12, 16, 20\}$

Reducing (mod $6$)

$S = \{0, 4, 2, 0, 4, 2\}$

Not distinct.

Common factor 2.

Can't be 1.

No inverse.

For $x = 5$ and $m = 6$.

$S = \{0 \cdot 5, 1 \cdot 5, 2 \cdot 5, 3 \cdot 5, 4 \cdot 5, 5 \cdot 5\}$

$= \{0, 5, 4, 3, 2, 1\}$

All distinct,

contains 1!

5 is multiplicative inverse of 5 (mod $6$).

(Hmm. What normal number is it own multiplicative inverse?) 1

Multiply both sides by 5.

$x = 15 = 3 \pmod{6}$

$x = 3 \pmod{6}$

4 $x = 2 \pmod{6}$

Two solutions!

x = 2, 5 \pmod{6}

Very different for elements with inverses.
Thm: If gcd\( (x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

Proof Sketch: The set \( S = \{0x, 1x, \ldots, (m - 1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).
Proof review. Consequence.

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... For \( x = 4 \) and \( m = 6 \). All products of 4...
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\[ S = \]
Proof review. Consequence.

**Thm:** If $\gcd(x, m) = 1$, then $x$ has a multiplicative inverse modulo $m$.

**Proof Sketch:** The set $S = \{0x, 1x, \ldots, (m - 1)x\}$ contains $y \equiv 1 \mod m$ if all distinct modulo $m$.

... 

For $x = 4$ and $m = 6$. All products of $4$...

$S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\}$
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m-1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

... 

For \( x = 4 \) and \( m = 6 \). All products of 4...

\[ S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\} \]
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

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...  
For \( x = 4 \) and \( m = 6 \). All products of 4...

\[
S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}
\]

reducing \( \pmod{6} \)
Proof review. Consequence.

**Thm:** If $\gcd(x, m) = 1$, then $x$ has a multiplicative inverse modulo $m$.

**Proof Sketch:** The set $S = \{0x, 1x, \ldots, (m - 1)x\}$ contains $y \equiv 1 \mod m$ if all distinct modulo $m$.

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reducing (mod 6)

$S = \{0, 4, 2, 0, 4, 2\}$
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reducing (mod 6)

$S = \{0, 4, 2, 0, 4, 2\}$

Not distinct.
Proof review. Consequence.

**Thm:** If $\gcd(x, m) = 1$, then $x$ has a multiplicative inverse modulo $m$.

**Proof Sketch:** The set $S = \{0x, 1x, \ldots, (m - 1)x\}$ contains $y \equiv 1 \pmod{m}$ if all distinct modulo $m$.

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reducing (mod 6)

$S = \{0, 4, 2, 0, 4, 2\}$

Not distinct. Common factor 2.
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m-1)x\} \) contains \( y \equiv 1 \pmod{m} \) if all distinct modulo \( m \).

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S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}
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reducing \( \pmod{6} \)

\[
S = \{0, 4, 2, 0, 4, 2\}
\]

Not distinct. Common factor 2. Can’t be 1.
Proof review. Consequence.

Thm: If gcd\((x, m)\) = 1, then \(x\) has a multiplicative inverse modulo \(m\).

Proof Sketch: The set \(S = \{0x, 1x, \ldots, (m-1)x\}\) contains \(y \equiv 1 \mod m\) if all distinct modulo \(m\).

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For \(x = 4\) and \(m = 6\). All products of 4...
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S = \{0, 4, 2, 0, 4, 2\}
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Reducing \((\text{mod } 6)\)

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For \( x = 5 \) and \( m = 6 \).
Thm: If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

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For \( x = 4 \) and \( m = 6 \). All products of 4...
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S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}
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reducing (mod 6)
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S = \{0, 4, 2, 0, 4, 2\}
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Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).
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S =
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**Thm:** If $\gcd(x, m) = 1$, then $x$ has a multiplicative inverse modulo $m$.

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- $S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}$
- reducing (mod 6)
- $S = \{0, 4, 2, 0, 4, 2\}$

Not distinct. Common factor 2. Can’t be 1. No inverse.

For $x = 5$ and $m = 6$.

- $S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\}$
**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

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Reducing \( \mod 6 \)
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S = \{0, 4, 2, 0, 4, 2\}
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Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).
\[
S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m - 1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

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reducing \( \mod 6 \)
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Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).
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reducing \( \mod 6 \)

\[
S = \{0, 4, 2, 0, 4, 2\}
\]

Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).

\[
S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}
\]

All distinct,
**Proof review. Consequence.**

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m-1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

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reducing \( \pmod{6} \)

\[
S = \{0, 4, 2, 0, 4, 2\}
\]

Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).

\[
S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}
\]

All distinct, contains 1!
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m-1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

... For \( x = 4 \) and \( m = 6 \). All products of 4...

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S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}
\]
reducing \( \mod 6 \)

\[
S = \{0, 4, 2, 0, 4, 2\}
\]
Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).

\[
S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}
\]
All distinct, contains 1! 5 is multiplicative inverse of 5 \( \mod 6 \).
Proof review. Consequence.

**Thm:** If $\gcd(x, m) = 1$, then $x$ has a multiplicative inverse modulo $m$.

**Proof Sketch:** The set $S = \{0x, 1x, \ldots, (m-1)x\}$ contains $y \equiv 1 \mod m$ if all distinct modulo $m$.

For $x = 4$ and $m = 6$. All products of $4$...

$S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}$

Reducing (mod 6)

$S = \{0, 4, 2, 0, 4, 2\}$

Not distinct. Common factor 2. Can’t be 1. No inverse.

For $x = 5$ and $m = 6$.

$S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}$

All distinct, contains 1! 5 is multiplicative inverse of 5 (mod 6).

(Hmm. What normal number is it own multiplicative inverse?)
Proof review. Consequence.

Thm: If $\gcd(x, m) = 1$, then $x$ has a multiplicative inverse modulo $m$.

Proof Sketch: The set $S = \{0x, 1x, \ldots, (m - 1)x\}$ contains $y \equiv 1 \mod m$ if all distinct modulo $m$.

... For $x = 4$ and $m = 6$. All products of 4:

$S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}$

reducing (mod 6)

$S = \{0, 4, 2, 0, 4, 2\}$

Not distinct. Common factor 2. Can’t be 1. No inverse.

For $x = 5$ and $m = 6$.

$S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}$

All distinct, contains 1! 5 is multiplicative inverse of 5 (mod 6). (Hmm. What normal number is it own multiplicative inverse?) 1
Proof review. Consequence.

**Thm:** If \( \text{gcd}(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m-1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

... For \( x = 4 \) and \( m = 6 \). All products of 4...
\[
S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}
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reducing \( \mod 6 \)
\[
S = \{0, 4, 2, 0, 4, 2\}
\]
Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).
\[
S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}
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All distinct, contains 1! 5 is multiplicative inverse of 5 \( \mod 6 \).
(Hmm. What normal number is it own multiplicative inverse?) 1 -1.
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m - 1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

For \( x = 4 \) and \( m = 6 \). All products of 4...

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reducing \( \pmod{6} \)

\[ S = \{0, 4, 2, 0, 4, 2\} \]

Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).

\[ S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\} \]

All distinct, contains 1! 5 is multiplicative inverse of 5 \( \pmod{6} \).

(Hmm. What normal number is it own multiplicative inverse?) 1 -1.

\[ 5x = 3 \pmod{6} \]
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m-1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

... 

For \( x = 4 \) and \( m = 6 \). All products of 4...

\[
S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}
\]

reducing \( \pmod{6} \)

\[
S = \{0, 4, 2, 0, 4, 2\}
\]

Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).

\[
S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}
\]

All distinct, contains 1! 5 is multiplicative inverse of 5 \( \pmod{6} \).

(Hmm. What normal number is it own multiplicative inverse?) 1 \(-1\).

\[
5x = 3 \pmod{6} \text{ What is } x?
\]
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m - 1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

... 
For \( x = 4 \) and \( m = 6 \). All products of 4...
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S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}
\]
reducing \((\mod 6)\)
\[
S = \{0, 4, 2, 0, 4, 2\}
\]
Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).
\[
S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}
\]
All distinct, contains 1! 5 is multiplicative inverse of 5 \((\mod 6)\).
(Hmm. What normal number is it own multiplicative inverse?) 1 -1.

\[
5x = 3 \mod 6 \] What is \( x \)? Multiply both sides by 5.
Thm: If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

Proof Sketch: The set \( S = \{0x, 1x, \ldots, (m - 1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

For \( x = 4 \) and \( m = 6 \). All products of 4...
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S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}
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reducing \( \mod 6 \)
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S = \{0, 4, 2, 0, 4, 2\}
\]
Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).
\[
S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}
\]
All distinct, contains 1! 5 is multiplicative inverse of 5 \( \mod 6 \).
(Hmm. What normal number is it own multiplicative inverse?) 1 -1.

\[
5x = 3 \mod 6 \]
What is \( x \)? Multiply both sides by 5.
\[
x = 15
\]
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m - 1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

... For \( x = 4 \) and \( m = 6 \). All products of 4...

\[
S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}
\]

reducing \((\mod 6)\)

\[
S = \{0, 4, 2, 0, 4, 2\}
\]

Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).

\[
S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}
\]

All distinct, contains 1! 5 is multiplicative inverse of 5 \((\mod 6)\). (Hmm. What normal number is it own multiplicative inverse?) 1 -1.

\[
5x = 3 \mod 6 \text{ What is } x? \text{ Multiply both sides by } 5.
\]

\[
x = 15 = 3 \mod 6
\]
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m-1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

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\[ S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\} \]

reducing (mod 6)

\[ S = \{0, 4, 2, 0, 4, 2\} \]

Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).

\[ S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\} \]

All distinct, contains 1! 5 is multiplicative inverse of 5 (mod 6).

(Hmm. What normal number is it own multiplicative inverse?) 1 -1.

\[ 5x = 3 \mod 6 \] What is \( x \)? Multiply both sides by 5.

\[ x = 15 = 3 \mod 6 \]

\[ 4x = 3 \mod 6 \]
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m - 1)x\} \) contains 
\( y \equiv 1 \mod m \) if all distinct modulo \( m \).

... For \( x = 4 \) and \( m = 6 \). All products of 4...
\( S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\} \)
reducing \( \mod 6 \)
\( S = \{0, 4, 2, 0, 4, 2\} \)
Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).
\( S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\} \)
All distinct, contains 1! 5 is multiplicative inverse of 5 \( \mod 6 \).  
(Hmm. What normal number is it own multiplicative inverse?) 1 -1.

\[ 5x = 3 \mod 6 \] What is \( x \)? Multiply both sides by 5.
\[ x = 15 = 3 \mod 6 \]

\[ 4x = 3 \mod 6 \] No solutions.
Proof review. Consequence.

**Thm:** If gcd\((x, m)\) = 1, then \(x\) has a multiplicative inverse modulo \(m\).

**Proof Sketch:** The set \(S = \{0x, 1x, \ldots, (m−1)x\}\) contains \(y \equiv 1 \mod m\) if all distinct modulo \(m\).

... For \(x = 4\) and \(m = 6\). All products of 4...

\[S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}\]

reducing (mod 6)

\[S = \{0, 4, 2, 0, 4, 2\}\]

Not distinct. Common factor 2. Can’t be 1. No inverse.

For \(x = 5\) and \(m = 6\).

\[S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}\]

All distinct, contains 1! 5 is multiplicative inverse of 5 (mod 6).

(Hmm. What normal number is it own multiplicative inverse?) 1 -1.

\[5x = 3 \pmod{6}\] What is \(x\)? Multiply both sides by 5.

\[x = 15 = 3 \pmod{6}\]

\[4x = 3 \pmod{6}\] No solutions. Can’t get an odd.
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m-1)x\} \) contains \( y \equiv 1 \pmod{m} \) if all distinct modulo \( m \).

\[ ...\]

For \( x = 4 \) and \( m = 6 \). All products of 4...
\[
S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}
\]
rereducing (mod 6)
\[
S = \{0, 4, 2, 0, 4, 2\}
\]
Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).
\[
S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}
\]
All distinct, contains 1! 5 is multiplicative inverse of 5 (mod 6).

(Hmm. What normal number is it own multiplicative inverse?) 1 -1.

\[
5x = 3 \pmod{6} \text{ What is } x? \text{ Multiply both sides by } 5.
\]
\[
x = 15 = 3 \pmod{6}
\]
\[
4x = 3 \pmod{6} \text{ No solutions. Can’t get an odd.}
\]
\[
4x = 2 \pmod{6}
\]
Proof review. Consequence.

**Thm:** If \(\gcd(x, m) = 1\), then \(x\) has a multiplicative inverse modulo \(m\).

**Proof Sketch:** The set \(S = \{0x, 1x, \ldots, (m-1)x\}\) contains \(y \equiv 1 \mod m\) if all distinct modulo \(m\).

... For \(x = 4\) and \(m = 6\). All products of 4...

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S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}
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reducing \((\mod 6)\)

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S = \{0, 4, 2, 0, 4, 2\}
\]

Not distinct. Common factor 2. Can’t be 1. No inverse.

For \(x = 5\) and \(m = 6\).

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S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}
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All distinct, contains 1! 5 is multiplicative inverse of 5 \((\mod 6)\). (Hmm. What normal number is it own multiplicative inverse?) 1 -1.

\[
5x = 3 \pmod{6} \quad \text{What is } x? \quad \text{Multiply both sides by 5.}
\]

\[
x = 15 = 3 \pmod{6}
\]

\[
4x = 3 \pmod{6} \quad \text{No solutions. Can’t get an odd.}
\]

\[
4x = 2 \pmod{6} \quad \text{Two solutions!}
\]
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m - 1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

For \( x = 4 \) and \( m = 6 \). All products of 4...

\[ S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\} \]

Reducing \((\mod 6)\)

\[ S = \{0, 4, 2, 0, 4, 2\} \]

Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).

\[ S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\} \]

All distinct, contains 1! 5 is multiplicative inverse of 5 \((\mod 6)\).

(Hmm. What normal number is it own multiplicative inverse?) 1 -1.

\[ 5x = 3 \mod 6 \] What is \( x \)? Multiply both sides by 5.

\[ x = 15 = 3 \mod 6 \]

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Thm: If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

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Very different for elements with inverses.
If \( \gcd(x,m) = 1 \).

One to one: there is a unique pre-image.

Onto: the sizes of the domain and co-domain are the same.

\[ f(x) = \begin{cases} 3 & \text{if } x = 3 \text{,} \\ 6 & \text{if } x = 2 \text{,} \\ 1 & \text{if } x = 3 \mod 4 \end{cases} \]

\[ f(0) = 0. \]

All the images are distinct.

\[ \Rightarrow \text{unique pre-image for any image.} \]

\[ x = 2, \ m = 4. \]

\[ f(1) = 2, \ f(2) = 0, \ f(3) = 2. \]

\[ f(0) = 0. \]

Not a bijection.
Proof Review 2: Bijections.

If \( \gcd(x,m) = 1 \).

Then the function \( f(a) = xa \mod m \) is a bijection.
Proof Review 2: Bijections.

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Proof Review 2: Bijectons.

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\( x = 3, m = 4 \).
\( f(1) = 3(1) = 3 \mod 4 \),
Proof Review 2: Bijections.

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One to one: there is a unique pre-image.
Onto: the sizes of the domain and co-domain are the same.

\( x = 3, m = 4 \).

\[ f(1) = 3(1) = 3 \ (\text{mod} \ 4), \ f(2) = 6 = 2 \ (\text{mod} \ 4), \]
Proof Review 2: Bijections.

If $\gcd(x,m) = 1$.

Then the function $f(a) = xa \mod m$ is a bijection.

One to one: there is a unique pre-image.

Onto: the sizes of the domain and co-domain are the same.

$x = 3, m = 4$.

$f(1) = 3(1) = 3 \pmod{4}, f(2) = 6 = 2 \pmod{4}, f(3) = 1 \pmod{3}$. 

Oh yeah.

$f(0) = 0$.

Bijection $\equiv$ unique pre-image and same size.

All the images are distinct. $\Rightarrow$ unique pre-image for any image.
If \( \gcd(x,m) = 1 \).
Then the function \( f(a) = xa \mod m \) is a bijection.
One to one: there is a unique pre-image.
Onto: the sizes of the domain and co-domain are the same.
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Bijection $\equiv$ unique pre-image and same size.
    All the images are distinct. $\implies$ unique pre-image for any image.
Proof Review 2: Bijects.

If \( \gcd(x,m) = 1 \).

Then the function \( f(a) = xa \mod m \) is a bijection.

One to one: there is a unique pre-image.

Onto: the sizes of the domain and co-domain are the same.

\( x = 3, m = 4 \).

\[ f(1) = 3(1) = 3 \mod 4, \quad f(2) = 6 = 2 \mod 4, \quad f(3) = 1 \mod 3. \]

Oh yeah. \( f(0) = 0 \).

Bijection \( \equiv \) unique pre-image and same size.

All the images are distinct. \( \implies \) unique pre-image for any image.

\( x = 2, m = 4 \).
Proof Review 2: Bijections.

If \( \gcd(x,m) = 1 \).

Then the function \( f(a) = xa \mod m \) is a bijection.

One to one: there is a unique pre-image.

Onto: the sizes of the domain and co-domain are the same.

\( x = 3, m = 4 \).

\( f(1) = 3(1) = 3 \mod 4, f(2) = 6 = 2 \mod 4, f(3) = 1 \mod 3 \).

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\( x = 2, m = 4 \).

\( f(1) = 2, f(2) = 0, f(3) = 2 \)

Oh yeah. \( f(0) = 0 \).

Not a bijection.
Finding inverses.

How to find the inverse?

Algorithm:
Try all numbers up to $x$ to see if it divides both $x$ and $m$.

Very slow.
Finding inverses.

How to find the inverse?
How to find if $x$ has an inverse modulo $m$?
Finding inverses.

How to find the inverse?
How to find if \( x \) has an inverse modulo \( m \)?
Find \( \text{gcd} \ (x, m) \).
Finding inverses.

How to find the inverse?
How to find if $x$ has an inverse modulo $m$?
Find $\gcd(x, m)$.
   Greater than 1?
Finding inverses.

How to find the inverse?

How to find if \( x \) has an inverse modulo \( m \)?

Find gcd (\( x, m \)).

- Greater than 1? No multiplicative inverse.
Finding inverses.

How to find the inverse?

How to find if \( x \) has an inverse modulo \( m \)?

Find \( \text{gcd} (x, m) \).
- Greater than 1? No multiplicative inverse.
- Equal to 1?
Finding inverses.

How to find the inverse?
How to find if \( x \) has an inverse modulo \( m \)?

Find gcd \( (x, m) \).
- Greater than 1? No multiplicative inverse.
- Equal to 1? Mutliplicative inverse.
Finding inverses.

How to find the inverse?

How to find if \( x \) has an inverse modulo \( m \)?

Find \( \text{gcd} (x, m) \).
- Greater than 1? No multiplicative inverse.
- Equal to 1? Multiplicative inverse.

Algorithm:
Finding inverses.

How to find the inverse?

How to find if $x$ has an inverse modulo $m$?

Find \( \gcd(x, m) \).
- Greater than 1? No multiplicative inverse.
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Finding inverses.

How to find the inverse?

How to find if $x$ has an inverse modulo $m$?

Find $\text{gcd} \ (x, m)$.
  
  Greater than 1? No multiplicative inverse.
  
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Very slow.
Finding inverses.

How to find the inverse?
How to find if $x$ has an inverse modulo $m$?

Find gcd $(x, m)$.
   Greater than 1? No multiplicative inverse.
   Equal to 1? Multiplicative inverse.

Algorithm: Try all numbers up to $x$ to see if it divides both $x$ and $m$.
Very slow.
Inverses

Next up.
Inverses

Next up.
Inverses

Next up.

Euclid’s Algorithm.
Inverses

Next up.

Euclid’s Algorithm.
Runtime.
Inverses

Next up.

Euclid’s Algorithm.
Runtime.
Euclid’s Extended Algorithm.
Does 2 have an inverse mod 8?

No. Any multiple of 2 is 2 away from 0 + 8k for any k ∈ N.

Does 2 have an inverse mod 9?

Yes. 5

2(5) = 10 = 1 mod 9.

Does 6 have an inverse mod 9?

No. Any multiple of 6 is 3 away from 0 + 9k for any k ∈ N.

3 = gcd(6, 9)!

x has an inverse modulo m if and only if gcd(x, m) > 1?

No. gcd(x, m) = 1?

Yes.

Now what?:

Compute gcd!

Compute Inverse modulo m.
Does 2 have an inverse mod 8? No.
Does 2 have an inverse mod 8? No.
    Any multiple of 2 is 2 away from $0 + 8k$ for any $k \in \mathbb{N}$. 

Does 2 have an inverse mod 9? Yes.
    $5 \cdot 2 \equiv 10 \equiv 1 \pmod{9}$.

Does 6 have an inverse mod 9? No.
    Any multiple of 6 is 3 away from $0 + 9k$ for any $k \in \mathbb{N}$.

$x$ has an inverse modulo $m$ if and only if $\gcd(x, m) = 1$?
Does 2 have an inverse mod 8? No.
Any multiple of 2 is 2 away from 0 + 8k for any \( k \in \mathbb{N} \).

Does 2 have an inverse mod 9?
Does 2 have an inverse mod 8? No.
   Any multiple of 2 is 2 away from $0 + 8k$ for any $k \in \mathbb{N}$.
Does 2 have an inverse mod 9? Yes.
Does 2 have an inverse mod 8? No.
Any multiple of 2 is 2 away from $0 + 8k$ for any $k \in \mathbb{N}$.

Does 2 have an inverse mod 9? Yes. 5
Does 2 have an inverse mod 8? No.
    Any multiple of 2 is 2 away from $0 + 8k$ for any $k \in \mathbb{N}$.
Does 2 have an inverse mod 9? Yes. 5
Does 2 have an inverse mod 8? No. Any multiple of 2 is 2 away from $0 + 8k$ for any $k \in \mathbb{N}$.

Does 2 have an inverse mod 9? Yes. 5

$2(5) = 10 \equiv 1 \mod 9$. 

$\gcd(6, 9) = 3 \neq 1$? No.

$\gcd(x, m) = 1$? Yes. Now what?: Compute gcd! Compute Inverse modulo $m$. 
Does 2 have an inverse mod 8? No. Any multiple of 2 is 2 away from $0 + 8k$ for any $k \in \mathbb{N}$.

Does 2 have an inverse mod 9? Yes. 5
\[ 2(5) = 10 = 1 \mod 9. \]

Does 6 have an inverse mod 9?

\[ 3 = \gcd(6, 9) \neq 1 \]

No. $\gcd(x, m) > 1$?
Does 2 have an inverse mod 8? No.
   Any multiple of 2 is 2 away from 0 + 8k for any \( k \in \mathbb{N} \).

Does 2 have an inverse mod 9? Yes. 5
   \( 2(5) = 10 = 1 \mod 9. \)

Does 6 have an inverse mod 9? No.
Does 2 have an inverse mod 8? No.
    Any multiple of 2 is 2 away from $0 + 8k$ for any $k \in \mathbb{N}$.

Does 2 have an inverse mod 9? Yes. 5
    $2(5) = 10 = 1 \mod 9$.

Does 6 have an inverse mod 9? No.
    Any multiple of 6 is 3 away from $0 + 9k$ for any $k \in \mathbb{N}$. 

\[ \gcd(6, 9) \neq 1 \] 

x has an inverse modulo m if and only if \( \gcd(x, m) = 1 \).
Does 2 have an inverse mod 8? No.
    Any multiple of 2 is 2 away from $0 + 8k$ for any $k \in \mathbb{N}$.

Does 2 have an inverse mod 9? Yes. 5
    $2(5) = 10 = 1 \mod 9$.

Does 6 have an inverse mod 9? No.
    Any multiple of 6 is 3 away from $0 + 9k$ for any $k \in \mathbb{N}$.
    $3 = \gcd(6, 9)$!
Does 2 have an inverse mod 8? No.
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\[
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Does 6 have an inverse mod 9? No.
    Any multiple of 6 is 3 away from 0 + 9k for any \( k \in \mathbb{N} \).
    3 = gcd(6, 9)!

\( x \) has an inverse modulo \( m \) if and only if
Does 2 have an inverse mod 8? No.
   Any multiple of 2 is 2 away from $0 + 8k$ for any $k \in \mathbb{N}$.

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   $2(5) = 10 = 1 \mod 9$.

Does 6 have an inverse mod 9? No.
   Any multiple of 6 is 3 away from $0 + 9k$ for any $k \in \mathbb{N}$.
   $3 = \gcd(6, 9)!

x has an inverse modulo m if and only if
   $\gcd(x, m) > 1$?
Does 2 have an inverse mod 8? No.
Any multiple of 2 is 2 away from $0 + 8k$ for any $k \in \mathbb{N}$.

Does 2 have an inverse mod 9? Yes. $5$
$2(5) = 10 = 1 \mod 9$.

Does 6 have an inverse mod 9? No.
Any multiple of 6 is 3 away from $0 + 9k$ for any $k \in \mathbb{N}$.
$3 = \gcd(6, 9)!$

$x$ has an inverse modulo $m$ if and only if
$\gcd(x, m) > 1$? No.
$\gcd(x, m) = 1$?
Does 2 have an inverse mod 8? No.
   Any multiple of 2 is 2 away from 0 + 8k for any \( k \in \mathbb{N} \).

Does 2 have an inverse mod 9? Yes. 5
   \( 2(5) = 10 = 1 \mod 9 \).

Does 6 have an inverse mod 9? No.
   Any multiple of 6 is 3 away from 0 + 9k for any \( k \in \mathbb{N} \).
   \( 3 = gcd(6, 9) \)!

\( x \) has an inverse modulo \( m \) if and only if
   \( gcd(x, m) > 1 \)? No.
   \( gcd(x, m) = 1 \)? Yes.
Does 2 have an inverse mod 8? No.
   Any multiple of 2 is 2 away from 0 + 8k for any \( k \in \mathbb{N} \).

Does 2 have an inverse mod 9? Yes. 5
   \( 2(5) = 10 = 1 \mod 9 \).

Does 6 have an inverse mod 9? No.
   Any multiple of 6 is 3 away from 0 + 9k for any \( k \in \mathbb{N} \).
   \( 3 = \gcd(6, 9) \)

\( x \) has an inverse modulo \( m \) if and only if
   \( \gcd(x, m) > 1 ? \) No.
   \( \gcd(x, m) = 1 ? \) Yes.

Now what?:
   Compute \( \gcd \)!
Does 2 have an inverse mod 8? No.
   Any multiple of 2 is 2 away from $0 + 8k$ for any $k \in \mathbb{N}$.

Does 2 have an inverse mod 9? Yes. 5
   $2(5) = 10 = 1 \mod 9$.

Does 6 have an inverse mod 9? No.
   Any multiple of 6 is 3 away from $0 + 9k$ for any $k \in \mathbb{N}$.
   $3 = \gcd(6, 9)$!

$x$ has an inverse modulo $m$ if and only if
   $\gcd(x, m) > 1$? No.
   $\gcd(x, m) = 1$? Yes.

Now what?:
   Compute gcd!
   Compute Inverse modulo $m$. 

\[ \frac{26}{30} \]
Does 2 have an inverse mod 8? No.
Any multiple of 2 is 2 away from \(0 + 8k\) for any \(k \in \mathbb{N}\).

Does 2 have an inverse mod 9? Yes. 5
\[2(5) = 10 = 1 \mod 9.\]

Does 6 have an inverse mod 9? No.
Any multiple of 6 is 3 away from \(0 + 9k\) for any \(k \in \mathbb{N}\).
\[3 = \gcd(6, 9)\]

\(x\) has an inverse modulo \(m\) if and only if
\[\gcd(x, m) > 1? \text{ No.}\]
\[\gcd(x, m) = 1? \text{ Yes.}\]

Now what?:
Compute \(\gcd\)!
Compute Inverse modulo \(m\).
Divisibility...

Notation: $d|x$ means “$d$ divides $x$” or
Divisibility...

**Notation:** $d | x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$. 
Notation: $d | x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

Fact: If $d | x$ and $d | y$ then $d | (x + y)$ and $d | (x - y)$. 
Notation: $d|x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

Fact: If $d|x$ and $d|y$ then $d|(x + y)$ and $d|(x - y)$.

Is it a fact?
Notation: $d|x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

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Proof: $d | x$ and $d | y$ or
Notation: $d|x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

Fact: If $d|x$ and $d|y$ then $d|(x + y)$ and $d|(x - y)$.

Is it a fact? Yes? No?

Proof: $d|x$ and $d|y$ or $x = \ell d$ and $y = kd$
Notation: $d|x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

Fact: If $d|x$ and $d|y$ then $d|(x + y)$ and $d|(x - y)$.

Is it a fact? Yes? No?

Proof: $d|x$ and $d|y$ or $x = \ell d$ and $y = kd$

$\Rightarrow x - y = kd - \ell d$
Notation: $d|x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

Fact: If $d|x$ and $d|y$ then $d|(x + y)$ and $d|(x - y)$.

Is it a fact? Yes? No?

Proof: $d|x$ and $d|y$ or

$x = \ell d$ and $y = kd$

$\implies x - y = kd - \ell d = (k - \ell)d$
Notation: $d|\,x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

Fact: If $d|\,x$ and $d|\,y$ then $d|\,(x + y)$ and $d|\,(x - y)$.

Is it a fact? Yes? No?

Proof: $d|\,x$ and $d|\,y$ or $x = \ell d$ and $y = kd$

$\implies x - y = kd - \ell d = (k - \ell)d \implies d|(x - y)$
Notation: \( d \mid x \) means “\( d \) divides \( x \)” or \( x = kd \) for some integer \( k \).

Fact: If \( d \mid x \) and \( d \mid y \) then \( d \mid (x + y) \) and \( d \mid (x - y) \).

Is it a fact? Yes? No?

Proof: \( d \mid x \) and \( d \mid y \) or
\[
x = \ell d \quad \text{and} \quad y = kd
\]
\[
\implies x - y = kd - \ell d = (k - \ell)d \implies d \mid (x - y)
\]
More divisibility

**Notation:** \( d \mid x \) means “\( d \) divides \( x \)” or \( x = kd \) for some integer \( k \).

**Lemma 1:** If \( d \mid x \) and \( d \mid y \) then \( d \mid y \) and \( d \mid \text{mod}(x, y) \).

**Proof:**

\[
\text{mod}(x, y) = x - \lfloor x/y \rfloor \cdot y = x - \lfloor s \rfloor \cdot y = \left(kd - s \ell \right)d
\]

Therefore \( d \mid \text{mod}(x, y) \).

And \( d \mid y \) since it is in condition.

**Lemma 2:** If \( d \mid y \) and \( d \mid \text{mod}(x, y) \) then \( d \mid y \) and \( d \mid x \).

**Proof:** Similar.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \text{mod}(x, y)) \).

**Proof:** \( x \) and \( y \) have the same set of common divisors as \( x \) and \( \text{mod}(x, y) \) by Lemma 1 and 2. Same common divisors \( \Rightarrow \) largest is the same.
More divisibility

**Notation:** $d | x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$. 

Lemma 1: If $d | x$ and $d | y$ then $d | y$ and $d | \text{mod}(x, y)$.

Proof: 

\[
\text{mod}(x, y) = x - \lfloor \frac{x}{y} \rfloor \cdot y = x - \lfloor s \rfloor \cdot y
\]

for integer $s = kd - \ell d$ for integers $k, \ell$ where $x = kd$ and $y = \ell d = (k - s \ell) d$. 

Therefore $d | \text{mod}(x, y)$.

And $d | y$ since it is in condition.

Lemma 2: If $d | y$ and $d | \text{mod}(x, y)$ then $d | y$ and $d | x$.

Proof: Similar.

GCD Mod Corollary: 

$\gcd(x, y) = \gcd(y, \text{mod}(x, y))$.

Proof: $x$ and $y$ have same set of common divisors as $x$ and $\text{mod}(x, y)$ by Lemma 1 and 2. 

Same common divisors $\Rightarrow$ largest is the same.
More divisibility

**Notation:** $d|x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d|x$ and $d|y$ then $d|y$ and $d| \text{mod}(x, y)$.
More divisibility

Notation: $d | x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

Lemma 1: If $d | x$ and $d | y$ then $d | y$ and $d | \text{mod}(x, y)$.

Proof:
$$\text{mod}(x, y) = x - \lfloor x/y \rfloor \cdot y$$
More divisibility

**Notation:** \( d \mid x \) means “\( d \) divides \( x \)” or \( x = kd \) for some integer \( k \).

**Lemma 1:** If \( d \mid x \) and \( d \mid y \) then \( d \mid y \) and \( d \mid \text{mod} \ (x, y) \).

**Proof:**
\[
\text{mod} \ (x, y) = x - \left\lfloor \frac{x}{y} \right\rfloor \cdot y = x - \left\lfloor s \right\rfloor \cdot y \quad \text{for integer} \ s
\]
More divisibility

**Notation:** $d|x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d|x$ and $d|y$ then $d|y$ and $d| \text{mod} (x, y)$.

**Proof:**
\[
\text{mod} (x, y) = x - \lfloor x/y \rfloor \cdot y
\]
\[
= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s
\]
\[
= kd - sℓd \quad \text{for integers } k, ℓ \text{ where } x = kd \text{ and } y = ℓd
\]

**GCD Mod Corollary:**
\[
\gcd(x, y) = \gcd(y, \text{mod} (x, y))
\]

**Proof:** $x$ and $y$ have the same set of common divisors as $x$ and $\text{mod} (x, y)$ by Lemma 1 and 2. The same common divisors $\Rightarrow$ largest is the same.
More divisibility

**Notation:** $d | x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d | x$ and $d | y$ then $d | y$ and $d | \text{mod} (x, y)$.

**Proof:**

\[
\text{mod} (x, y) = x - \lfloor x/y \rfloor \cdot y \\
= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s \\
= kd - s \ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d \\
= (k - s \ell)d
\]
More divisibility

**Notation:** $d \mid x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d \mid x$ and $d \mid y$ then $d \mid y$ and $d \mid \text{mod}(x, y)$.

**Proof:**
\[
\text{mod}(x, y) = x - \lfloor x/y \rfloor \cdot y
\]
\[
= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s
\]
\[
= kd - s\ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d
\]
\[
= (k - s\ell)d
\]
Therefore $d \mid \text{mod}(x, y)$. 

**Lemma 2:** If $d \mid y$ and $d \mid \text{mod}(x, y)$ then $d \mid y$ and $d \mid x$.

**Proof...:** Similar.

**GCD Mod Corollary:** $\gcd(x, y) = \gcd(y, \text{mod}(x, y))$.

**Proof:** $x$ and $y$ have same set of common divisors as $x$ and $\text{mod}(x, y)$ by Lemma 1 and 2. Same common divisors $\Rightarrow$ largest is the same.
More divisibility

**Notation:** $d \mid x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d \mid x$ and $d \mid y$ then $d \mid y$ and $d \mid \text{mod} (x, y)$.

**Proof:**

\[
\text{mod} (x, y) = x - \left\lfloor \frac{x}{y} \right\rfloor \cdot y \\
= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s \\
= kd - s\ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d \\
= (k - s\ell)d
\]

Therefore $d \mid \text{mod} (x, y)$. And $d \mid y$ since it is in condition.

**Lemma 2:**

If $d \mid y$ and $d \mid \text{mod} (x, y)$ then $d \mid y$ and $d \mid x$.

**Proof...:** Similar.
More divisibility

**Notation:** $d|x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d|x$ and $d|y$ then $d|y$ and $d|\mod(x, y)$.

**Proof:**

\[
\mod(x, y) = x - \lfloor x/y \rfloor \cdot y
\]

\[
= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s
\]

\[
= kd - s\ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d
\]

\[
= (k - s\ell)d
\]

Therefore $d|\mod(x, y)$. And $d|y$ since it is in condition.
More divisibility

**Notation:** \( d | x \) means “\( d \) divides \( x \)” or \( x = kd \) for some integer \( k \).

**Lemma 1:** If \( d | x \) and \( d | y \) then \( d | y \) and \( d | \text{mod} (x, y) \).

**Proof:**
\[
\text{mod} (x, y) = x - \lfloor x/y \rfloor \cdot y
\]
\[
= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s
\]
\[
= kd - s\ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d
\]
\[
= (k - s\ell)d
\]
Therefore \( d | \text{mod} (x, y) \). And \( d | y \) since it is in condition. \( \square \)

**Lemma 2:** If \( d | y \) and \( d | \text{mod} (x, y) \) then \( d | y \) and \( d | x \).

**Proof:** Similar.
More divisibility

**Notation:** \(d | x\) means “\(d\) divides \(x\)” or \(x = kd\) for some integer \(k\).

**Lemma 1:** If \(d | x\) and \(d | y\) then \(d | y\) and \(d | \text{mod} (x, y)\).

**Proof:**
\[
\text{mod} (x, y) = x - \lfloor x/y \rfloor \cdot y \\
= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s \\
= kd - s \ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d \\
= (k - s \ell)d
\]
Therefore \(d | \text{mod} (x, y)\). And \(d | y\) since it is in condition.

**Lemma 2:** If \(d | y\) and \(d | \text{mod} (x, y)\) then \(d | y\) and \(d | x\).

**Proof...:** Similar. Try this at home.
More divisibility

**Notation:** $d | x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d | x$ and $d | y$ then $d | y$ and $d | \text{mod} (x, y)$.

**Proof:**
\[
\text{mod} (x, y) = x - \lfloor x/y \rfloor \cdot y
\]
\[
= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s
\]
\[
= kd - s \ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d
\]
\[
= (k - s \ell)d
\]
Therefore $d | \text{mod} (x, y)$. And $d | y$ since it is in condition.

**Lemma 2:** If $d | y$ and $d | \text{mod} (x, y)$ then $d | y$ and $d | x$.

**Proof...:** Similar. Try this at home.
More divisibility

**Notation:** \( d \mid x \) means “\( d \) divides \( x \)” or \( x = kd \) for some integer \( k \).

**Lemma 1:** If \( d \mid x \) and \( d \mid y \) then \( d \mid y \) and \( d \mid \text{mod} \ (x, y) \).

**Proof:**

\[
\text{mod} \ (x, y) = x - \lfloor x/y \rfloor \cdot y
= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s
= kd - s\ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d
= (k - s\ell)d
\]

Therefore \( d \mid \text{mod} \ (x, y) \). And \( d \mid y \) since it is in condition.

**Lemma 2:** If \( d \mid y \) and \( d \mid \text{mod} \ (x, y) \) then \( d \mid y \) and \( d \mid x \).

**Proof...:** Similar. Try this at home.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \ \text{mod} \ (x, y)) \).
More divisibility

**Notation:** $d | x$ means “$d$ divides $x$” or
$x = kd$ for some integer $k$.

**Lemma 1:** If $d | x$ and $d | y$ then $d | y$ and $d | \mod (x, y)$.

**Proof:**
$$
\mod (x, y) = x - \lfloor x/y \rfloor \cdot y
= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s
= kd - s \ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d
= (k - s \ell)d
$$

Therefore $d | \mod (x, y)$. And $d | y$ since it is in condition.

**Lemma 2:** If $d | y$ and $d | \mod (x, y)$ then $d | y$ and $d | x$.

**Proof...:** Similar. Try this at home.

**GCD Mod Corollary:** $\gcd(x, y) = \gcd(y, \mod (x, y))$.

**Proof:** $x$ and $y$ have **same** set of common divisors as $x$ and $\mod (x, y)$ by Lemma 1 and 2.
More divisibility

Notation: $d|x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

Lemma 1: If $d|x$ and $d|y$ then $d|y$ and $d| \text{mod} (x, y)$.

Proof:
\[
\text{mod} (x, y) = x - \lfloor x/y \rfloor \cdot y
\]
\[
= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s
\]
\[
= kd - s \ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d
\]
\[
= (k - s \ell )d
\]

Therefore $d| \text{mod} (x, y)$. And $d|y$ since it is in condition.

Lemma 2: If $d|y$ and $d| \text{mod} (x, y)$ then $d|y$ and $d|x$.

Proof...: Similar. Try this at home.

GCD Mod Corollary: $\gcd(x, y) = \gcd(y, \text{mod} (x, y))$.

Proof: $x$ and $y$ have same set of common divisors as $x$ and $\text{mod} (x, y)$ by Lemma 1 and 2.

Same common divisors $\implies$ largest is the same.
More divisibility

**Notation:** \( d | x \) means “\( d \) divides \( x \)” or \( x = kd \) for some integer \( k \).

**Lemma 1:** If \( d | x \) and \( d | y \) then \( d | y \) and \( d | \text{mod} \ (x, y) \).

**Proof:**
\[
\text{mod} \ (x, y) = x - \lfloor x/y \rfloor \cdot y \\
= x - [s] \cdot y \quad \text{for integer } s \\
= kd - s\ell d \quad \text{for integers } k, \ell \quad \text{where } x = kd \text{ and } y = \ell d \\
= (k - s\ell)d
\]

Therefore \( d | \text{mod} \ (x, y) \). And \( d | y \) since it is in condition.

**Lemma 2:** If \( d | y \) and \( d | \text{mod} \ (x, y) \) then \( d | y \) and \( d | x \).

**Proof...:** Similar. Try this at home.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \text{mod} \ (x, y)) \).

**Proof:** \( x \) and \( y \) have **same** set of common divisors as \( x \) and \( \text{mod} \ (x, y) \) by Lemma 1 and 2.

Same common divisors \( \implies \) largest is the same.
Euclid’s algorithm.

**GCD Mod Corollary:** \( \text{gcd}(x, y) = \text{gcd}(y, \mod(x, y)) \).
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)?
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7
Euclid’s algorithm.

**GCD Mod Corollary:** $\text{gcd}(x, y) = \text{gcd}(y, \text{mod}(x, y))$.

Hey, what’s $\text{gcd}(7, 0)$? 7 since 7 divides 7 and 7 divides 0
Euclid’s algorithm.

**GCD Mod Corollary**: \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)?
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)?  \( x \)
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)? \( x \)

```scheme
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))) ***
```
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)?  \( x \)

(\text{define (euclid x y)}
 (if (= y 0)
   x
   (euclid y (mod x y))))

**Theorem:** \((\text{euclid x y}) = \gcd(x, y)\) if \( x \geq y \).
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)?  \( x \)

(define (euclid x y)
  (if (= y 0)
    x
    (euclid y (mod x y))))  ***

**Theorem:** \( (\text{euclid } x \ y) = \gcd(x, y) \) if \( x \geq y \).

**Proof:** Use Strong Induction.
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)? x

(\text{define (euclid x y)}
  (\text{if (= y 0)}
    x
    (euclid y (mod x y))))) ***

**Theorem:** (euclid x y) = \( \gcd(x, y) \) if \( x \geq y \).

**Proof:** Use Strong Induction.
**Base Case:** \( y = 0 \), “\( x \) divides \( y \) and \( x \)”
Euclid’s algorithm.

**GCD Mod Corollary:** $\gcd(x, y) = \gcd(y, \mod(x, y))$.

Hey, what’s $\gcd(7, 0)$? 7 since 7 divides 7 and 7 divides 0
What’s $\gcd(x, 0)$? $x$

(define (euclid x y)
  (if (= y 0)
    x
    (euclid y (mod x y))))

***

**Theorem:** $(\text{euclid } x y) = \gcd(x, y)$ if $x \geq y$.

**Proof:** Use Strong Induction.

**Base Case:** $y = 0$, “$x$ divides $y$ and $x$”

$\implies$ “$x$ is common divisor and clearly largest.”
Euclid’s algorithm.

**GCD Mod Corollary:** \( \text{gcd}(x, y) = \text{gcd}(y, \mod(x, y)) \).

Hey, what’s \( \text{gcd}(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \text{gcd}(x, 0) \)?  \( x \)

```
(define (euclid x y)
  (if (= y 0)
    x
    (euclid y (mod x y)))
```

**Theorem:** \( (\text{euclid } x \ y) = \text{gcd}(x, y) \) if \( x \geq y \).

**Proof:** Use Strong Induction.

**Base Case:** \( y = 0 \), “\( x \) divides \( y \) and \( x \)”

\[ \implies “x \) is common divisor and clearly largest.”

**Induction Step:** \( \mod(x, y) < y \leq x \) when \( x \geq y \)
Euclid’s algorithm.

**GCD Mod Corollary:** \( \text{gcd}(x, y) = \text{gcd}(y, \mod (x, y)) \).

Hey, what’s \( \text{gcd}(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \text{gcd}(x, 0) \)? \( x \)

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))  ***

**Theorem:** \( \text{(euclid } x \ y) = \text{gcd}(x, y) \) if \( x \geq y \).

**Proof:** Use Strong Induction.

**Base Case:** \( y = 0 \), “\( x \) divides \( y \) and \( x \)”
  \( \implies \) “\( x \) is common divisor and clearly largest.”

**Induction Step:** \( \mod (x, y) < y \leq x \) when \( x \geq y \)

call in line (***) meets conditions plus arguments “smaller”
Euclid’s algorithm.

**GCD Mod Corollary:** \( \text{gcd}(x, y) = \text{gcd}(y, \mod (x, y)) \).

Hey, what’s gcd(7, 0)? 7 since 7 divides 7 and 7 divides 0
What’s gcd(x, 0)? x

(define (euclid x y)
  (if (= y 0)
    x
    (euclid y (mod x y))))

***

**Theorem:** \( (\text{euclid} \ x \ y) = \text{gcd}(x, y) \) if \( x \geq y \).

**Proof:** Use Strong Induction.

**Base Case:** \( y = 0 \), “\( x \) divides \( y \) and \( x \)”

\[ \Rightarrow \text{“} x \text{ is common divisor and clearly largest.”} \]

**Induction Step:** \( \mod (x, y) < y \leq x \) when \( x \geq y \)

call in line (***).meets conditions plus arguments “smaller”
and by strong induction hypothesis
Euclid’s algorithm.

**GCD Mod Corollary:** $\gcd(x, y) = \gcd(y, \mod(x, y))$.

Hey, what’s $\gcd(7, 0)$? 7 since 7 divides 7 and 7 divides 0
What’s $\gcd(x, 0)$? $x$

(define (euclid x y)
  (if (= y 0)
    x
    (euclid y (mod x y)))) ***

**Theorem:** $(\text{euclid } x y) = \gcd(x, y)$ if $x \geq y$.

**Proof:** Use Strong Induction.

**Base Case:** $y = 0$, “$x$ divides $y$ and $x$”
  $\implies$ “$x$ is common divisor and clearly largest.”

**Induction Step:** $\mod(x, y) < y \leq x$ when $x \geq y$

call in line (***), meets conditions plus arguments “smaller”
  and by strong induction hypothesis
computes $\gcd(y, \mod(x, y))$
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \text{mod}(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)? \( x \)

(define (euclid x y)
  (if (= y 0)
    x
    (euclid y (mod x y)))) ***

**Theorem:** \((\text{euclid } x \ y) = \gcd(x, \ y) \) if \( x \geq y \).

**Proof:** Use Strong Induction.

**Base Case:** \( y = 0 \), “\( x \) divides \( y \) and \( x \)”
\( \implies \) “\( x \) is common divisor and clearly largest.”

**Induction Step:** \( \text{mod}(x, y) < y \leq x \) when \( x \geq y \)

call in line (****) meets conditions plus arguments “smaller”
and by strong induction hypothesis
computes \( \gcd(y, \text{mod}(x, y)) \)
which is \( \gcd(x, y) \) by GCD Mod Corollary.
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)? \( x \)

```scheme
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))***
```

**Theorem:** \( (euclid x y) = \gcd(x, y) \) if \( x \geq y \).

**Proof:** Use Strong Induction.

**Base Case:** \( y = 0 \), “\( x \) divides \( y \) and \( x \)”

\[ \implies \text{“} x \text{ is common divisor and clearly largest.”} \]

**Induction Step:** \( \mod(x, y) < y \leq x \) when \( x \geq y \)

Call in line (***)) meets conditions plus arguments “smaller”

and by strong induction hypothesis
computes \( \gcd(y, \mod(x, y)) \)
which is \( \gcd(x, y) \) by GCD Mod Corollary.
Modular Arithmetic Lecture in a minute.

Modular Arithmetic: \( x \equiv y \ (\text{mod } N) \) if \( x = y + kN \) for some integer \( k \).
Modular Arithmetic Lecture in a minute.

Modular Arithmetic: \( x \equiv y \pmod{N} \) if \( x = y + kN \) for some integer \( k \).

For \( a \equiv b \pmod{N} \), and \( c \equiv d \pmod{N} \),
\( ac = bd \pmod{N} \) and \( a + b = c + d \pmod{N} \).
Modular Arithmetic Lecture in a minute.

Modular Arithmetic: \( x \equiv y \pmod{N} \) if \( x = y + kN \) for some integer \( k \).

For \( a \equiv b \pmod{N} \), and \( c \equiv d \pmod{N} \),
\[ ac = bd \pmod{N} \] and \( a + b = c + d \pmod{N} \).

Division?
Modular Arithmetic Lecture in a minute.

Modular Arithmetic: \( x \equiv y \pmod{N} \) if \( x = y + kN \) for some integer \( k \).

For \( a \equiv b \pmod{N} \), and \( c \equiv d \pmod{N} \),
\[ ac = bd \pmod{N} \]
and \( a + b = c + d \pmod{N} \).

Division? Multiply by multiplicative inverse.
Modular Arithmetic Lecture in a minute.

Modular Arithmetic: \( x \equiv y \pmod{N} \) if \( x = y + kN \) for some integer \( k \).

For \( a \equiv b \pmod{N} \), and \( c \equiv d \pmod{N} \),
\[ ac = bd \pmod{N} \] and \( a + b = c + d \pmod{N} \).

Division? Multiply by multiplicative inverse.
\( a \pmod{N} \) has multiplicative inverse, \( a^{-1} \pmod{N} \).
Modular Arithmetic: $x \equiv y \pmod{N}$ if $x = y + kN$ for some integer $k$.

For $a \equiv b \pmod{N}$, and $c \equiv d \pmod{N}$,
\[ac = bd \pmod{N}\] and \[a + b = c + d \pmod{N} \]

Division? Multiply by multiplicative inverse.
\[a \pmod{N}\] has multiplicative inverse, $a^{-1} \pmod{N}$.
If and only if $\gcd(a, N) = 1$. 
Modular Arithmetic Lecture in a minute.

Modular Arithmetic: $x \equiv y \pmod{N}$ if $x = y + kN$ for some integer $k$.

For $a \equiv b \pmod{N}$, and $c \equiv d \pmod{N}$,  
$ac \equiv bd \pmod{N}$ and $a + b \equiv c + d \pmod{N}$.

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Why? If: $f(x) = ax \pmod{N}$ is a bijection on $\{1, \ldots, N - 1\}$. 
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Why? If: \( f(x) = ax \pmod{N} \) is a bijection on \( \{1, \ldots, N-1\} \).

\[ ax - ay = 0 \pmod{N} \implies a(x - y) \text{ is a multiple of } N. \]
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If \( \gcd(a, N) = 1 \),
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Modular Arithmetic Lecture in a minute.

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$ax - ay = 0 \pmod{N} \implies a(x - y)$ is a multiple of $N$.
If $gcd(a, N) = 1$,
then $(x - y)$ must contain all primes in prime factorization of $N$, and is therefore be bigger than $N$. 
Modular Arithmetic Lecture in a minute.

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Only if: For $a = xd$ and $N = yd$,
any $ma + kN = d(mx - ky)$ or is a multiple of $d$,
and is not 1.