"Theorem": All students love CS70.

"Proof": Let $P(n)$ be "given a set of $n$ students, they all love CS70".

Base case: $P(0)$ is trivially true.

Inductive step:
Assume $P(n)$ is true. Suppose we're given a set of students $\{S_1, S_2, ..., S_n, S_{n+1}\}$.

By inductive hypothesis, students in $\{S_1, ..., S_n\}$ all love CS70. Similarly, students in $\{S_2, ..., S_{n+1}\}$ all love CS70.

$\Rightarrow S_1, ..., S_{n+1}$ all love CS70.

By the principle of induction, since $P(0)$ is true and \(\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1)\), we know the statement is true.

Q: Do you agree? What's wrong?

$P(0) \not\Rightarrow P(1)$

So we didn't really prove "$\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1)$".
# 1. Induction

Induction is a technique for proving \( \forall n \in \mathbb{N} \), \( P(n) \).

## 1.1 Simple Induction

To prove \( P(n) \) is true for all \( n \in \mathbb{N} \), use

**Base case:** check \( P(0) \) holds

**Inductive Step:** Show \( P(k) \Rightarrow P(k+1) \) for all \( k \in \mathbb{N} \).

E.g. Prove that \( \sum_{j=0}^{n} a_j = \frac{ar^{n+1} - a}{r-1} \) where \( a, r \in \mathbb{R}, r \neq 1, n \in \mathbb{N} \).

**Proof:** Let \( P(n) \) be the statement \( \sum_{j=0}^{n} a_j = \frac{ar^{n+1} - a}{r-1} \).

**Base case:** \( P(0) \) holds, because \( \sum_{j=0}^{0} a_j = a = \frac{ar - a}{r-1} \).

**Inductive step:** \( \forall k \in \mathbb{N}, P(k) \Rightarrow P(k+1) \).

**Inductive hypothesis:** Assume \( P(k) \) is true, i.e. assume \( \sum_{j=0}^{k} a_j = \frac{ar^{k+1} - a}{r-1} \).

Want to show \( \sum_{j=0}^{k+1} a_j = \frac{ar^{k+2} - a}{r-1} \).

\[
\sum_{j=0}^{k+1} a_j = \sum_{j=0}^{k} a_j + a_{k+1}
\]

\[
= \frac{ar^{k+1} - a}{r-1} + a_{k+1}
\]

\[
= \frac{ar^{k+1} - a}{r-1} + \frac{ar^{k+2} - ar^{k+1}}{r-1}
\]

\[
= \frac{ar^{k+2} - a}{r-1}
\]

\( \Box \)
E.g. Prove that $2^n < n!$ for every integer $n \geq 4$.

**Pf:** Let $P(n)$ be the statement $2^n < n!$.

- **Base Case:** $P(4)$ holds, because $2^4 = 16 < 4! = 24$.
- **Inductive Step:** WTS $\forall k \in \mathbb{N}$, $k \geq 4$, $P(k) \Rightarrow P(k+1).

Assume $P(k)$, i.e. assume $2^k < k!$ for some integer $k \geq 4$.

Want to show $2^{k+1} < (k+1)! = k! (k+1)$.

Notice that

$$2^{k+1} = 2^k \times 2$$

$$< k! \times 2$$

$$< k! \times (k+1)$$

$$= (k+1)!$$

E.g. Prove a map with $n$ lines is 2-colorable, where $n \in \mathbb{N}$.

**Pf:** Let $P(n)$ be the statement a map with $n$ lines is 2-colorable.

- **Base Case:** $P(0)$ holds, because we can color the entire map using a single color.

- **Inductive Step:** WTS $\forall k \in \mathbb{N}$, $P(k) \Rightarrow P(k+1)$.

Assume $P(k)$, i.e. assume a map with $k$ lines is
Given a map with $k+l$ lines, remove a line $\ell$. Do it so that the new map is $2$-colorable. 

Consider a map with $k+l$ lines. By IH, the new map is $2$-colorable.

Add line $\ell$. Pick one side and swap the colors.

The result is still $2$-colorable, because for each shared border, it's either $\ell$ or not $\ell$.

1. If $\ell$ is not $\ell$, two sides have different colors by IH.
2. If $\ell = \ell$, two sides now have different colors because of the swap.

Want to show a map with $k+l$ lines is $2$-colorable.

Inductive Step: Assume the result holds for some $k \in \mathbb{Z}^+$, i.e., $\sum_{x=1}^{x=k} (2x-1) = m^2$ for some $m \in \mathbb{Z}^+$.

Want to show $\sum_{x=1}^{x=k+1} (2x-1) = n^2$ for some $n \in \mathbb{Z}^+$.

Proof: Let $P(k)$ be the statement $\sum_{x=1}^{x=k} (2x-1) = m^2$, for some $m \in \mathbb{Z}^+$.

E.g. Prove the sum of the first $k$ odd numbers is a perfect square.

$$\sum_{x=1}^{x=k} (2x-1) = m^2$$
\[ 1=1, \quad 1+3=4=2^2, \quad 1+3+5=9=3^2, \quad \ldots \]

Let \( P(n) \) be the statement \( \sum_{x=1}^{n} (2x-1) = n^2 \).

**Base case:** \( P(1) \) holds because \( 1 = 1^2 \).

**Inductive Step:** Assume \( \sum_{x=1}^{k} (2x-1) = k^2 \) for some \( k \in \mathbb{Z}^+ \).

Then \( \sum_{x=1}^{k+1} (2x-1) = \left( \sum_{x=1}^{k} (2x-1) \right) + 2k+1 \)

\[ = k^2 + 2k + 1 \]

\[ = (k+1)^2. \]

\[ \square \]

### 1.2 Strong Induction

To prove \( P(n) \) is true for all \( n \in \mathbb{N} \), use

**Base case:** check \( P(0) \) holds.

**Inductive Step:** Show \( \forall k \in \mathbb{N}, \left[ P(0) \land \ldots \land P(k) \right] \Rightarrow P(k+1) \).

**E.g.** Prove that if \( n \) is an integer greater than 1, then \( n \) can be written as a product of primes.

**Pf:**

**Base case:** \( 2 \) is a prime and a product of itself, so the statement holds for \( n = 2 \).

**Inductive Step:** Assume all integers \( 2 \leq j \leq k \) can be written as a product of primes.

Consider \( k+1 \). If \( k+1 \) is prime, we’re done.

Otherwise, \( k+1 = ab \) for some integers \( a, b \) with \( 2 \leq a, b < k+1 \).
By IH, a·b can be written as a product of primes. Hence, k+1 can be written as a product of primes.

\[ \forall n \geq 12, P(n). \]

E.g. Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

**Pf:** Let \( P(n) \) be the statement \( n = 4x + 5y \) for some \( x, y \in \mathbb{N} \).

- **Base case:** \( P(12) \) is true \( 12 = 4 \cdot 3 + 5 \cdot 0 \).
- \( P(13), P(14), P(15) \) holds because ...

- **Inductive step:** Assume \( P(n) \) holds for all \( 12 \leq n \leq k \) for some \( k \geq 15 \).

Consider \( k+1 \)

- If \( (k+1) - 4 = 4x + 5y \), \( k+1 = 4(x+1) + 5y \).
  - But need \( (k+1) - 4 \geq 12 \), i.e. \( k \geq 15 \)
  - Since \( (k+1) - 4 \geq 12 \), by IH, \( (k+1) - 4 = 4x + 5y \) for \( x, y \in \mathbb{N} \), so \( k+1 = 4(x+1) + 5y \).

Rem. **Well-ordering principle** states \( S \subseteq \mathbb{N}, S \neq \emptyset \), then \( S \) has a least element.

The validity of the principle of induction and strong induction follows from WOP.
To prove a statement holds for recursively defined objects, use

**Base case:** the result holds for all elements specified in the base case

**Recursive step:** show if the statement holds for each element used to construct new elements, then it holds for these new elements.

**E.g.** Binary trees can be constructed recursively.

Define height $h(T)$ recursively.

**Base case** ($T = \text{root}$): $h(T) = 0$

**Recursive step** ($T = T_1 \cdot T_2$): $h(T) = 1 + \max (h(T_1), h(T_2))$

Define number of vertices $n(T)$ recursively.

**Base case** ($T = \text{root}$): $n(T) = 1$.

**Recursive step** ($T = T_1 \cdot T_2$): $n(T) = 1 + n(T_1) + n(T_2)$.

Prove $n(T) \leq 2^{h(T)+1} - 1$ for any binary tree $T$.

**Pf:** **Base case:** $T = \text{root}$. Then $n(T) = 1$ and $h(T) = 0$. Statement holds because $1 \leq 2^{0+1} - 1 = 1$

**Recursive step:** Consider $T = T_1 \cdot T_2$. Want to show $n(T) \leq 2^{h(T)+1} - 1$.
Notice that
\[
n(T) \overset{\text{def}}{=} 1 + n(T_1) + n(T_2)
\]
\[
\leq 1 + (2^{h(T_1)} - 1) + (2^{h(T_2)} - 1)
\]
\[
= 2^{h(T_1)} + 2^{h(T_2)} - 1
\]
\[
\leq 2 \cdot \max \left( 2^{h(T_1)} + 1, 2^{h(T_2)} + 1 \right) - 1
\]
\[
= 2 \cdot 2^{\max (h(T_1), h(T_2)) + 1} - 1
\]
\[
= 2^{h(T)} - 1
\]
\[
= 2^{h(T) + 1} - 1.
\]